DIFFERENTIAL AND CONSTANT-BEAMWIDTH BEAMFORMING WITH UNIFORM RECTANGULAR ARRAYS

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ABSTRACT

This paper presents a differential and constant-beamwidth (CB) beamforming approach, which is based on Kronecker-product (KP) beamforming with uniform rectangular arrays (URAs). We decompose a global rectangular beamformer into a KP of two sub-beamformers: a constant-beamwidth beamformer along the y-axis and a super-directive (SD) beamformer along the x-axis. We propose two design methods to derive global rectangular beamformers whose either white noise gain (WNG) or directivity factor (DF) may be set by design. We show that the proposed rectangular beamformers exhibit improved directivity with respect to the linear CB beamformer, particularly in high frequencies, and improved robustness to spatially white noise with respect to the linear SD beamformer. Finally, the proposed rectangular beamformers exhibit the constant-beamwidth property, with an inherent tradeoff between the constant-beamwidth threshold frequency and the array directivity, which is tuned by the number of microphones along each axis.

***Index Terms*—** Microphone arrays, constant-beamwidth beamformer, differential beamforming, uniform rectangular arrays.

1. INTRODUCTION

Beamforming design in the frequency domain has been an active area of research in the past few decades, aiming to exploit spatial information to retrieve signals of interest while attenuating undesirable background noise [1–3]. Most commonly in the literature, beamforming is designed and applied with uniform linear arrays (ULAs) due to their simplicity and easy-to-analyze nature [4]. Nevertheless, they suffer from inherent drawbacks. For example, the phase difference between every two adjacent microphones is identical. This implies that a ULA may only sense the desired signal from a single perspective.

To enrich the beamforming sensing perspective, more sophisticated array geometries have been proposed [5–10]. In particular, URAs have been shown valuable for direction of arrival (DOA) estimation methods [11, 12], and in the context of differential beamforming, that is, when the interelement spacing (along both axes) is small [13, 14]. These approaches either enable improvements in the array robustness to spatially white noise or allow high directivity beamforming even when the desired signal significantly deviates from the endfire direction.

The concept of KP beamforming has been extensively utilized in recent years. It allows a flexible design in which a global beamformer is decomposed into a KP of independent sub-beamformers that may be individually designed and optimized [15–19]. Each sub-beamformer may be optimized by a different criterion, yielding a global beamformer that is "optimized" according to all requirements. The portion of each optimization criterion is typically determined by the relative sizes of the corresponding sub-beamformers.

Broadband applications (e.g., involving communication and speech signals) tend to suffer from a frequency-varying spatial array response. In particular, the variance of the main-lobe beamwidth across different frequencies is usually of the highest interest. To minimize this variance, it is common to employ CB beamformers that maintain a fixed beamwidth over a wide frequency range, typically above a threshold frequency which is a function of the array aperture and the beamforming design technique [20–22].

In this paper, we present a differential and constant-beamwidth beamforming approach, which is based on KP beamforming with URAs. We decompose a global rectangular beamformer into a KP of two sub-beamformers: a CB beamformer along the y-axis and an SD beamformer along the x-axis. We propose two design methods to derive global rectangular beamformers whose either WNG or DF may be set by design. We show that the proposed rectangular beamformers exhibit improved directivity with respect to the linear CB beamformer, particularly in high frequencies, and improved robustness to spatially white noise with respect to the linear SD beamformer. Finally, the proposed rectangular beamformers exhibit the constant-beamwidth property, with an inherent tradeoff between the constant-beamwidth threshold frequency and the array directivity, which is tuned by the number of microphones along each axis.

2. SIGNAL MODEL

Consider a signal of interest propagating in the shape of a plane wave from the farfield in an anechoic acoustic environment at the speed of sound, i.e., $c = 340$ m/s, in an azimuth angle ϕ and an elevation angle θ. The plane wave impinges on a two-dimensional (2-D) microphone array located on the x-y plane which is composed of M_{x} and M_{y} omnidirectional microphones along the x-axis and y-axis, respectively. We denote the positions of the microphones by $(m_{\mathsf{x}}, m_{\mathsf{y}})$, with $m_{\mathsf{x}} = 1, 2, \ldots, M_{\mathsf{x}}$ and $m_{\mathsf{y}} = 1, 2, \ldots, M_{\mathsf{y}}$. Then, defining the microphone located in $(1, 1)$ as the origin of the Cartesian coordinate, the array steering vector of length $M_{\mathsf{x}} M_{\mathsf{y}}$ is expressed by [2]:

$$\begin{aligned}\mathbf{d}_{\theta,\phi}(\omega) &= \left[\begin{array}{ccc} B_{\theta,\phi,1}(\omega)\,\mathbf{a}_{\theta,\phi}^T(\omega) & \cdots & B_{\theta,\phi,M_{\mathsf{y}}}(\omega)\,\mathbf{a}_{\theta,\phi}^T(\omega)\end{array}\right]^T \\ &= \mathbf{b}_{\theta,\phi}(\omega) \otimes \mathbf{a}_{\theta,\phi}(\omega),\end{aligned} \quad (1)$$

where

$$\begin{aligned}\mathbf{a}_{\theta,\phi}(\omega) &= \left[\begin{array}{cccc} A_{\theta,\phi,1}(\omega) & A_{\theta,\phi,2}(\omega) & \cdots & A_{\theta,\phi,M_{\mathsf{x}}}(\omega)\end{array}\right]^T \\ &= \left[\begin{array}{cccc} 1 & e^{\jmath\varpi_{\theta,\phi,\mathsf{x}}(\omega)} & \ldots & e^{\jmath(M_{\mathsf{x}}-1)\varpi_{\theta,\phi,\mathsf{x}}(\omega)}\end{array}\right]^T\end{aligned} \quad (2)$$

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is the steering vector associated with the x axis,

$$\begin{aligned}\mathbf{b}_{\theta,\phi}(\omega) &= \begin{bmatrix} B_{\theta,\phi,1}(\omega) & B_{\theta,\phi,2}(\omega) & \cdots & B_{\theta,\phi,M_{\mathsf{y}}}(\omega) \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & e^{\jmath\varpi_{\theta,\phi,\mathsf{y}}(\omega)} & \cdots & e^{\jmath(M_{\mathsf{y}}-1)\varpi_{\theta,\phi,\mathsf{y}}(\omega)} \end{bmatrix}^T \end{aligned} \quad (3)$$

is the steering vector associated with the y axis,

$$\varpi_{\theta,\phi,\mathsf{x}}(\omega) = \frac{\omega\delta_{\mathsf{x}}\sin\theta\cos\phi}{c},$$
$$\varpi_{\theta,\phi,\mathsf{y}}(\omega) = \frac{\omega\delta_{\mathsf{y}}\sin\theta\sin\phi}{c},$$

the superscript ^{T} denotes the transpose operator, $\otimes$ is the KP operator, $\jmath = \sqrt{-1}$ is the imaginary unit, $\omega = 2\pi f$ is the angular frequency, and $f > 0$ is the temporal frequency.

Exploiting the steering vector in (1), the observed signal vector of length $M_{\mathsf{x}}M_{\mathsf{y}}$ of the URA can be expressed in the frequency domain as [4]:

$$\begin{aligned}\mathbf{y}(\omega) &= \begin{bmatrix} \mathbf{y}_1^T(\omega) & \mathbf{y}_2^T(\omega) & \cdots & \mathbf{y}_{M_{\mathsf{y}}}^T(\omega) \end{bmatrix}^T \\ &= \mathbf{x}(\omega) + \mathbf{v}(\omega) \\ &= \mathbf{d}_{\theta,\phi}(\omega)X(\omega) + \mathbf{v}(\omega), \end{aligned} \quad (4)$$

where $X(\omega)$ is the zero-mean desired source signal, $\mathbf{v}(\omega)$ is the zero-mean additive noise signal vector, and

$$\begin{aligned}\mathbf{y}_{m_{\mathsf{y}}}(\omega) &= \begin{bmatrix} Y_{m_{\mathsf{y}},1}(\omega) & Y_{m_{\mathsf{y}},2}(\omega) & \cdots & Y_{m_{\mathsf{y}},M_{\mathsf{x}}}(\omega) \end{bmatrix}^T \\ &= \mathbf{x}_{m_{\mathsf{y}}}(\omega) + \mathbf{v}_{m_{\mathsf{y}}}(\omega) \\ &= B_{\theta,\phi,m_{\mathsf{y}}}(\omega)\mathbf{a}_{\theta,\phi}(\omega)X(\omega) + \mathbf{v}_{m_{\mathsf{y}}}(\omega), \end{aligned} \quad (5)$$

for $m_{\mathsf{y}} = 1, 2, \ldots, M_{\mathsf{y}}$, is the observed signal vector of length M_{x} of the m_{y}th ULA parallel to the x axis. Denoting the desired signal incident angle by (θ_0, ϕ_0) and dropping the dependence on ω, (4) becomes:

$$\mathbf{y} = (\mathbf{b}_{\theta_0,\phi_0} \otimes \mathbf{a}_{\theta_0,\phi_0})X + \mathbf{v}, \quad (6)$$

where $\mathbf{b}_{\theta_0,\phi_0} \otimes \mathbf{a}_{\theta_0,\phi_0} = \mathbf{d}_{\theta_0,\phi_0}$ is the steering matrix at (θ_0, ϕ_0), and the covariance matrix of $\mathbf{y}$ is

$$\mathbf{\Phi}_{\mathbf{y}} = E\left(\mathbf{y}\mathbf{y}^H\right) = p_X\mathbf{d}_{\theta_0,\phi_0}\mathbf{d}_{\theta_0,\phi_0}^H + \mathbf{\Phi}_{\mathbf{v}}, \quad (7)$$

where $E(\cdot)$ denotes mathematical expectation, the superscript ^{H} is the conjugate-transpose operator, $p_X = E\left(|X|^2\right)$ is the variance of X, and $\mathbf{\Phi}_{\mathbf{v}} = E\left(\mathbf{v}\mathbf{v}^H\right)$ is the covariance matrix of $\mathbf{v}$. Assuming that the variance of the noise is approximately the same at all sensors, we can express (7) as

$$\mathbf{\Phi}_{\mathbf{y}} = p_X\mathbf{d}_{\theta_0,\phi_0}\mathbf{d}_{\theta_0,\phi_0}^H + p_V\mathbf{\Gamma}_{\mathbf{v}}, \quad (8)$$

where p_V is the variance of the noise at the reference microphone (i.e., the origin of the Cartesian coordinate system) and $\mathbf{\Gamma}_{\mathbf{v}} = \mathbf{\Phi}_{\mathbf{v}}/p_V$ is the pseudo-coherence matrix of the noise. From (8), we deduce that the input signal-to-noise ratio (SNR) is

$$\mathrm{iSNR} = \frac{\mathrm{tr}\left(p_X\mathbf{d}_{\theta_0,\phi_0}\mathbf{d}_{\theta_0,\phi_0}^H\right)}{\mathrm{tr}\left(p_V\mathbf{\Gamma}_{\mathbf{v}}\right)} = \frac{p_X}{p_V}, \quad (9)$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a square matrix.

3. KRONECKER-PRODUCT BEAMFORMING

We would like to design a global rectangular beamformer $\mathbf{f}$ of length $M_{\mathsf{x}}M_{\mathsf{y}}$ as a KP of two linear sub-beamformers designed with respect to each one of the axes of the URA. Hence, $\mathbf{f}$ is of the form:

$$\mathbf{f} = \mathbf{w} \otimes \mathbf{h}, \quad (10)$$

where $\mathbf{h}$ is a linear sub-beamformer of length M_{x} and $\mathbf{w}$ is a linear sub-beamformer of length M_{y}. Then, the beamformer output signal is

$$Z = \mathbf{f}^H\mathbf{y} = X_{\mathrm{fd}} + V_{\mathrm{rn}}, \quad (11)$$

where Z is the estimate of X,

$$X_{\mathrm{fd}} = \left(\mathbf{w}^H\mathbf{b}_{\theta_0,\phi_0}\right)\left(\mathbf{h}^H\mathbf{a}_{\theta_0,\phi_0}\right)X \quad (12)$$

is the filtered desired signal, and

$$V_{\mathrm{rn}} = (\mathbf{w} \otimes \mathbf{h})^H\mathbf{v} \quad (13)$$

is the residual noise. In addition, it is clear that a distortionless constraint is given by

$$\mathbf{h}^H\mathbf{a}_{\theta_0,\phi_0} = 1,\ \mathbf{w}^H\mathbf{b}_{\theta_0,\phi_0} = 1. \quad (14)$$

Next, we relate the most prominent performance measures corresponding to $\mathbf{f}$. The output SNR and the gain in SNR are, respectively,

$$\mathrm{oSNR}(\mathbf{f}) = \frac{p_X}{p_V} \times \frac{\left|\mathbf{f}^H\mathbf{d}_{\theta_0,\phi_0}\right|^2}{\mathbf{f}^H\mathbf{\Gamma}_{\mathbf{v}}\mathbf{f}}, \quad (15)$$

and

$$\mathcal{G}(\mathbf{f}) = \frac{\mathrm{oSNR}(\mathbf{f})}{\mathrm{iSNR}} = \frac{\left|\mathbf{f}^H\mathbf{d}_{\theta_0,\phi_0}\right|^2}{\mathbf{f}^H\mathbf{\Gamma}_{\mathbf{v}}\mathbf{f}}, \quad (16)$$

from which we deduce the WNG:

$$\begin{aligned}\mathcal{W}(\mathbf{f}) &= \frac{\left|\mathbf{f}^H\mathbf{d}_{\theta_0,\phi_0}\right|^2}{\mathbf{f}^H\mathbf{f}} \\ &= \frac{\left|\mathbf{w}^H\mathbf{b}_{\theta_0,\phi_0}\right|^2}{\mathbf{w}^H\mathbf{w}} \times \frac{\left|\mathbf{h}^H\mathbf{a}_{\theta_0,\phi_0}\right|^2}{\mathbf{h}^H\mathbf{h}} \\ &= \mathcal{W}(\mathbf{w}) \times \mathcal{W}(\mathbf{h}), \end{aligned} \quad (17)$$

and the DF:

$$\mathcal{D}(\mathbf{f}) = \frac{\left|\mathbf{f}^H\mathbf{d}_{\theta_0,\phi_0}\right|^2}{\mathbf{f}^H\mathbf{\Gamma}_{\mathrm{d}}\mathbf{f}}, \quad (18)$$

where $\mathbf{\Gamma}_{\mathrm{d}}$ is the pseudo-coherence matrix of the spherically isotropic (diffuse) noise field [1, 14].

We end by defining the beampattern by

$$\begin{aligned}\mathcal{B}_{\theta,\phi}(\mathbf{f}) &= \mathbf{f}^H\mathbf{d}_{\theta,\phi} \\ &= \left(\mathbf{w}^H\mathbf{b}_{\theta,\phi}\right)\left(\mathbf{h}^H\mathbf{a}_{\theta,\phi}\right) \\ &= \mathcal{B}_{\theta,\phi}(\mathbf{w})\mathcal{B}_{\theta,\phi}(\mathbf{h}), \end{aligned} \quad (19)$$

where $\mathcal{B}_{\theta,\phi}(\mathbf{w}) = \mathbf{w}^H\mathbf{b}_{\theta,\phi}$ may be seen as the beampattern of $\mathbf{w}$ and $\mathcal{B}_{\theta,\phi}(\mathbf{h}) = \mathbf{h}^H\mathbf{a}_{\theta,\phi}$ may be seen as the beampattern of $\mathbf{h}$.

4. OPTIMAL CONSTANT-BEAMWIDTH BEAMFORMING

Assume we are interested in deriving a rectangular version of the SD beamformer [4], which is not necessarily a KP beamformer. That is, we would like to solve

$$\min_{\mathbf{f}} \ \mathbf{f}^H \mathbf{\Gamma}_{\mathrm{d}} \mathbf{f} \quad \text{s. t.} \quad \mathbf{f}^H \mathbf{d}_{\theta_0,\phi_0} = 1, \tag{20}$$

whose solution is obtained by

$$\begin{aligned} \mathbf{f}_{\mathrm{SD}} &= \frac{\mathbf{\Gamma}_{\mathrm{d}}^{-1} \mathbf{d}_{\theta_0,\phi_0}}{\mathbf{d}_{\theta_0,\phi_0}^H \mathbf{\Gamma}_{\mathrm{d}}^{-1} \mathbf{d}_{\theta_0,\phi_0}} \\ &= \frac{\left[\Sigma_{p=1}^{M_{\mathsf{y}}-1} \mathbf{J}_{M_{\mathsf{y}},p} \otimes \mathbf{\Gamma}_{\mathrm{d},p}\right]^{-1} \mathbf{d}_{\theta_0,\phi_0}}{\mathbf{d}_{\theta_0,\phi_0}^H \mathbf{\Gamma}_{\mathrm{d}}^{-1} \mathbf{d}_{\theta_0,\phi_0}}, \end{aligned} \tag{21}$$

where $\mathbf{\Gamma}_{\mathrm{d,p}}$ is the pth $M_{\mathsf{x}} \times M_{\mathsf{x}}$ block in the top block row of $\mathbf{\Gamma}_{\mathrm{d}}$ and

$$\left(\mathbf{J}_{M_{\mathsf{y}},p}\right)_{ij} = \begin{cases} 1 & |i-j| = p \\ 0 & |i-j| \neq p \end{cases}, \tag{22}$$

is a binary matrix of size $M_{\mathsf{y}} \times M_{\mathsf{y}}$ with ones on the $-p$th and pth diagonals and zeros elsewhere. In particular, $\mathbf{J}_{M_{\mathsf{y}},0} = \mathbf{I}_{M_{\mathsf{y}}}$, which is the identity matrix of size $M_{\mathsf{y}} \times M_{\mathsf{y}}$. Now, assuming

$$\frac{\delta_{\mathsf{y}}}{\delta_{\mathsf{x}}} > M_{\mathsf{x}} - 1, \tag{23}$$

(21) may be approximated by

$$\begin{aligned} \mathbf{f}_{\mathrm{SD}} &\approx \kappa \left(\mathbf{I}_{M_{\mathsf{y}}} \otimes \mathbf{\Gamma}_{\mathrm{d},1}^{-1}\right) \mathbf{d}_{\theta_0,\phi_0} \\ &= \kappa \left(\mathbf{I}_{M_{\mathsf{y}}} \otimes \mathbf{\Gamma}_{\mathrm{d},1}^{-1}\right) \left(\mathbf{b}_{\theta_0,\phi_0} \otimes \mathbf{a}_{\theta_0,\phi_0}\right) \\ &= \kappa \left(\mathbf{I}_{M_{\mathsf{y}}} \mathbf{b}_{\theta_0,\phi_0}\right) \otimes \left(\mathbf{\Gamma}_{\mathrm{d},1}^{-1} \mathbf{a}_{\theta_0,\phi_0}\right) \\ &= \kappa \bar{\mathbf{w}}_{\mathrm{DS}} \otimes \bar{\mathbf{h}}_{\mathrm{SD}}, \end{aligned} \tag{24}$$

where κ constitutes a normalization factor, $\mathbf{\Gamma}_{\mathrm{d},1}$ is the top-left block of $\mathbf{\Gamma}_{\mathrm{d}}$ of size $M_{\mathsf{x}} \times M_{\mathsf{x}}$, $\bar{\mathbf{w}}_{\mathrm{DS}}$ is the (unnormalized) linear delay-and-sum (DS) beamformer which operates on the ULAs in the y-axis and $\bar{\mathbf{h}}_{\mathrm{SD}}$ is the (unnormalized) linear SD beamformer which operates on the ULAs in the x-axis. In addition, the condition in (23) implies that the latter should be designed as differential beamformers, that is, with a small interelement spacing δ_{x}, whereas the interelement spacing in the y-axis, δ_{y}, should be larger. We note that the approximation in (24) is particularly more accurate in higher frequencies.

Since the optimal rectangular SD beamformer can be decomposed into a KP of two linear beamformers, with merely one of which optimized with respect to the array directivity, we may adapt the complementary beamformer to attain another array attributes. For example, to obtain CB beamformers, $\bar{\mathbf{w}}_{\mathrm{DS}}$ may be replaced by the modified rectangular window of [20], however, $\bar{\mathbf{w}}$ may be designed in general as any of the linear window-based CB beamformers suggested in [21]. Assuming the desired signal (speaker) is located on the x-y plane in the endfire direction, i.e., $\theta_0 = \pi/2, \phi_0 = 0$, this implies that

$$\begin{aligned} \mathbf{f}_{\mathrm{SD/CB}} &= \kappa \mathbf{w}_{\mathrm{rect}} \otimes \bar{\mathbf{h}}_{\mathrm{SD}} \\ &= \frac{\mathbf{w}_{\mathrm{rect}} \otimes \left(\mathbf{\Gamma}_{\mathrm{d},1}^{-1} \mathbf{a}_{\pi/2,0}\right)}{\mathbf{a}_{\pi/2,0}^H \mathbf{\Gamma}_{\mathrm{d},1}^{-1} \mathbf{a}_{\pi/2,0}}, \end{aligned} \tag{25}$$

with $\mathbf{w}_{\mathrm{rect}}$ being the linear modified rectangular window-based CB beamformer of length M_{y} and

$$\mathbf{a}_{\pi/2,0} = \begin{bmatrix} 1 & e^{\jmath\omega\delta_{\mathsf{x}}/c} & \cdots & e^{\jmath\omega(M_{\mathsf{x}}-1)\delta_{\mathsf{x}}/c} \end{bmatrix}^T. \tag{26}$$

We refer to $\mathbf{f}_{\mathrm{SD/CB}}$ as the rectangular super-directive constant-beamwidth (SD-CB) beamformer.

In many cases, it is desirable to explicitly set either the WNG or DF of the global beamformer. Therefore, we take advantage of the KP beamforming structure and the approach suggested in [23] in the context of ULAs, and modify $\bar{\mathbf{h}}_{\mathrm{SD}}$ accordingly.

Let us start with the WNG measure and let $\mathcal{W}_0$ be a desirable frequency-dependent WNG value of the global beamformer. Exploiting [23], we define the super-directive beamformer by

$$\mathbf{h}_{\mathrm{SD},\epsilon} = \frac{\left[\mathbf{\Gamma}_{\mathrm{d},1,\epsilon}^{-1} + \alpha \mathbf{I}_{M_{\mathsf{x}}}\right] \mathbf{a}_{\pi/2,0}}{\mathbf{a}_{\pi/2,0}^H \left[\mathbf{\Gamma}_{\mathrm{d},1,\epsilon}^{-1} + \alpha \mathbf{I}_{M_{\mathsf{x}}}\right] \mathbf{a}_{\pi/2,0}}, \tag{27}$$

where

$$\alpha = \frac{\mathbf{a}_{\pi/2,0}^H \mathbf{\Gamma}_{\mathrm{d},1,\epsilon}^{-1} \mathbf{a}_{\pi/2,0}}{M_{\mathsf{x}}} \left[\sqrt{\frac{\bar{\mathcal{W}}_0}{M_{\mathsf{x}} - \bar{\mathcal{W}}_0}} \left|\tan\varphi_\epsilon\right| - 1\right], \tag{28}$$

$$\mathbf{\Gamma}_{\mathrm{d},1,\epsilon} = \mathbf{\Gamma}_{\mathrm{d},1} + \epsilon \mathbf{I}_{M_{\mathsf{x}}}, \tag{29}$$

with $\epsilon = 10^{-7}$ being a frequency-independent regularization factor,

$$\cos\varphi_\epsilon = \frac{\mathbf{a}_{\pi/2,0}^H \mathbf{\Gamma}_{\mathrm{d},1,\epsilon}^{-1} \mathbf{a}_{\pi/2,0}}{\sqrt{M_{\mathsf{x}}}\sqrt{\mathbf{a}_{\pi/2,0}^H \mathbf{\Gamma}_{\mathrm{d},1,\epsilon}^{-2} \mathbf{a}_{\pi/2,0}}}, \tag{30}$$

and $\bar{\mathcal{W}}_0$ is given by

$$\bar{\mathcal{W}}_0 = \mathcal{W}_0 / \mathcal{W}\left(\mathbf{w}_{\mathrm{rect}}\right). \tag{31}$$

Then, the WNG of

$$\mathbf{f}_{\mathrm{SD/CB},\epsilon} = \mathbf{w}_{\mathrm{rect}} \otimes \mathbf{h}_{\mathrm{SD},\epsilon}, \tag{32}$$

to which we refer as the rectangular SD-CB beamformer of the first kind, is guaranteed to be $\mathcal{W}_0$. Clearly, we have

$$\begin{aligned} \mathcal{W}_0 &\le \mathcal{W}\left(\mathbf{w}_{\mathrm{rect}}\right) \times \max \ \bar{\mathcal{W}}_0 \\ &= \mathcal{W}\left(\mathbf{w}_{\mathrm{rect}}\right) \times M_{\mathsf{x}} \\ &= \mathcal{W}_{\max,\mathbf{w}}, \end{aligned} \tag{33}$$

where

$$\bar{\mathcal{W}}_0 \le M_{\mathsf{x}}, \tag{34}$$

with its maximum obtained for $\alpha \longrightarrow \infty$ as $\mathbf{h}_{\mathrm{SD},\epsilon} \longrightarrow \mathbf{h}_{\mathrm{DS}}$, and $\mathbf{h}_{\mathrm{DS}}$ is the DS beamformer.

Similarly, we may wish to set a desirable DF level of the global beamformer. Noting that

$$\mathbf{f}_{\mathrm{SD/CB}}^H \mathbf{\Gamma}_{\mathrm{d}} \mathbf{f}_{\mathrm{SD/CB}} = \bar{\mathbf{h}}_{\mathrm{SD}}^H \mathbf{\Gamma}_{\mathrm{d},\mathbf{w}} \bar{\mathbf{h}}_{\mathrm{SD}} \tag{35}$$

where

$$\mathbf{\Gamma}_{\mathrm{d},\mathbf{w}} = \left(\mathbf{w}_{\mathrm{rect}} \otimes \mathbf{I}_{M_{\mathsf{x}}}\right)^H \mathbf{\Gamma}_{\mathrm{d}} \left(\mathbf{w}_{\mathrm{rect}} \otimes \mathbf{I}_{M_{\mathsf{x}}}\right), \tag{36}$$

we may substitute $\mathbf{\Gamma}_{\mathrm{d}}$ with $\mathbf{\Gamma}_{\mathrm{d},\mathbf{w}}$ in equations (57)-(59) in [23] to obtain $\widetilde{\mathbf{h}}_{\mathrm{SD},\epsilon}$. Then, it is straightforward to show that the DF of the rectangular SD-CB beamformer of the second kind, which is given by

$$\widetilde{\mathbf{f}}_{\mathrm{SD/CB},\epsilon} = \mathbf{w}_{\mathrm{rect}} \otimes \widetilde{\mathbf{h}}_{\mathrm{SD},\epsilon}, \tag{37}$$

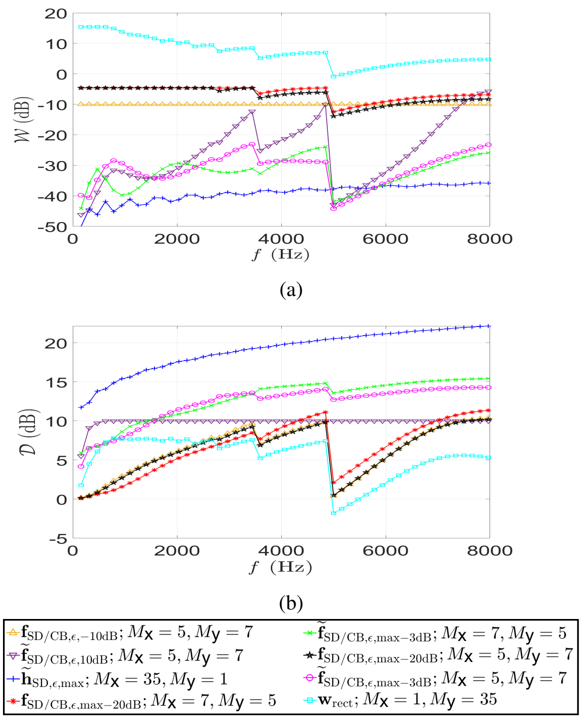

Fig. 1: WNG and DF measures for different values of $\mathcal{W}_0$, $\mathcal{D}_0$, M_{x} and M_{y} with $M=35$. (a) WNG and (b) DF.

equals a desirable frequency-dependent value $\mathcal{D}_0$. Clearly, we have

$$\mathcal{D}_0 \leq \left[\widetilde{\mathbf{h}}_{\mathrm{SD},\epsilon,\max}^H \mathbf{\Gamma}_{\mathrm{d},\mathbf{w}} \widetilde{\mathbf{h}}_{\mathrm{SD},\epsilon,\max}\right]^{-1} = \mathcal{D}_{\max,\mathbf{w}}, \tag{38}$$

where

$$\widetilde{\mathbf{h}}_{\mathrm{SD},\epsilon,\max} = \frac{\mathbf{\Gamma}_{\mathrm{d},\mathbf{w}}^{-1}\mathbf{a}_{\pi/2,0}}{\mathbf{a}_{\pi/2,0}^H \mathbf{\Gamma}_{\mathrm{d},\mathbf{w}}^{-1}\mathbf{a}_{\pi/2,0}}. \tag{39}$$

5. DESIGN EXAMPLES

Let us provide some design examples of $\widetilde{\mathbf{f}}_{\mathrm{SD/CB},\epsilon}$ and $\mathbf{f}_{\mathrm{SD/CB},\epsilon}$. According to the previous part, we first set the value of M_{y} and use [20] to obtain $\mathbf{w}_{\mathrm{rect}}$. Considering M as the total number of microphones in the URA, we immediately have $M_{\mathsf{x}} = M/M_{\mathsf{y}}$. Next, we may design either $\mathbf{h}_{\mathrm{SD},\epsilon}$ or $\widetilde{\mathbf{h}}_{\mathrm{SD},\epsilon}$, of length M_{x}, to set the either DF or the WNG of the URA. Clearly, with $\mathbf{f}_{\mathrm{SD/CB},\epsilon}$, as long as (33) is satisfied we may set $\mathcal{W}_0$ arbitrarily; and with $\widetilde{\mathbf{f}}_{\mathrm{SD/CB},\epsilon}$, as long as (38) is satisfied we may set $\mathcal{D}_0$ arbitrarily. Fig 1 demonstrates the WNG and DF for two different values of $\mathcal{W}_0$ and $\mathcal{D}_0$: $\mathcal{W}_0 = \{-10\mathrm{dB}, \mathcal{W}_{\max,\mathbf{w}} - 20\mathrm{dB}\}$, and $\mathcal{D}_0 = \{10\mathrm{dB}, \mathcal{D}_{\max,\mathbf{w}} - 3\mathrm{dB}\}$, with varying values of $M_{\mathsf{x}}M_{\mathsf{y}} = 35$ microphones, $\delta_{\mathsf{x}} = 5$ mm, $\delta_{\mathsf{y}} = 4$ cm and $\Delta_\phi = 40^o$. In terms of notations, $\mathbf{f}_{\mathrm{SD/CB},\epsilon,\max-20\mathrm{dB}}$, for example, stands for the $\mathbf{f}_{\mathrm{SD/CB},\epsilon}$ beamformer with $\mathcal{W}_0 = \mathcal{W}_{\max,\mathbf{w}} - 20\mathrm{dB}$.

To begin with, we note that the two rectangular beamformers whose WNG or DF is designed constant exhibit the desired value, with a performance deterioration in the complementary measure in edge frequencies in which the effective length of $\mathbf{w}_{\mathrm{rect}}$ drops [20].

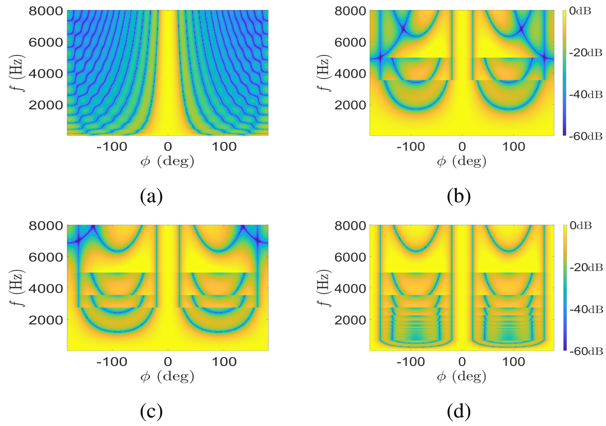

Fig. 2: Beampatterns as a function of the frequency and the azimuth angle ϕ for different values of M_{x} and M_{y} with $M=35$. (a) $\mathbf{h}_{\mathrm{SD},\epsilon,\max}$; $M_{\mathsf{x}} = 35, M_{\mathsf{y}} = 1$, (b) $\mathbf{f}_{\mathrm{SD/CB},\epsilon,\max-20\mathrm{dB}}$; $M_{\mathsf{x}} = 7, M_{\mathsf{y}} = 5$, (c) $\mathbf{f}_{\mathrm{SD/CB},\epsilon,\max-20\mathrm{dB}}$; $M_{\mathsf{x}} = 5, M_{\mathsf{y}} = 7$, and (d) $\mathbf{w}_{\mathrm{rect}}$; $M_{\mathsf{x}} = 1, M_{\mathsf{y}} = 35$.

Addressing the six remaining beamformers, we observe that the linear SD and CB beamformers exhibit superior DF and WNG performance, respectively. When both M_{x} and M_{y} are larger than 1 and with both types of the SD-CB beamformers, increasing M_{x} improves the DF in high frequencies but potentially worsens the DF in low frequencies. Addressing the WNG measure, the performance is improved upon increasing M_{x} in high frequencies with $\mathbf{f}_{\mathrm{SD/CB},\epsilon}$; in low frequencies and with $\widetilde{\mathbf{f}}_{\mathrm{SD/CB},\epsilon}$ there is no significant performance difference.

To investigate the constant-beamwidth property, we plot the beampatterns of four presented beamformers in Fig 2. It is evident that increasing M_{y} lowers the threshold frequency above which the constant-beamwidth property is obtained. In addition, we note that in frequencies larger than $f = 5$ kHz and $M_{\mathsf{y}} > 1$ the side lobes are considerably amplified. This is a by-product of the effective length drop of the linear modified rectangular window beamformer. We infer that the higher M_{x}, the better the DF in high frequencies, whereas the higher M_{y}, the lower the constant-beamwidth threshold frequency.

6. CONCLUSIONS

We have presented a differential and CB beamforming approach with URAs. Assuming the ratio between the interelement spacing along the y and x axes is larger than the number of microphones along the x-axis, the rectangular SD beamformer is shown to be approximated by a KP of a linear DS beamformer along the y-axis and a linear SD beamformer along the x-axis. Then, we replace the DS beamformer with a CB beamformer and propose two design methods to derive global rectangular beamformers whose either WNG or DF may be set by design. The proposed beamformers exhibit improved directivity with respect to the linear CB beamformer, particularly in high frequencies, and improved robustness to spatially white noise with respect to the linear SD beamformer. Finally, the proposed rectangular beamformers exhibit the constant-beamwidth property, with an inherent tradeoff between the constant-beamwidth threshold frequency and the array directivity, tuned by the number of microphones along each axis.

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