Contents

Preface *iii*

2 **Single-Channel Signal Enhancement in the Time Domain** *1*

3 **Single-Channel Signal Enhancement in the Frequency Domain** *27*

4 **Multichannel Signal Enhancement in the Time Domain** *47*

5 **Multichannel Signal Enhancement in the Frequency Domain** *75*

6 **An Exhaustive Class of Linear Filters** *123*

7 **Fixed Beamforming** *145*

8 **Adaptive Beamforming** *169*

9 **Differential Beamforming** *199*

10 **Beampattern Design** *221*

11 **Beamforming in the Time Domain** *237*

Preface

This manual provides solutions to the 250 problems in the textbook *Fundamentals of Signal Enhancement and Array Signal Processing* (J. Benesty, I. Cohen, and J. Chen, Wiley-IEEE Press, Singapore, 2018. ISBN: 978-1-119-29312-5).

The solutions were developed by the most excellent teaching assistant, undergraduate students, and graduate students, while taking the course *Spatial Signal Processing* (course number 046743) in the spring semester of 2019, which was taught by Prof. Israel Cohen at the Faculty of Electrical Engineering, Technion–Israel Institute of Technology.

We thank all of those who have made tremendous work and contributed solutions to this manual. This includes the outstanding teaching assistant, who developed solutions for three chapters, and managed the solutions by the students in the remaining chapters: **Or Yair**; the top graduate students: **Ronald Gold, Amir Ivry, Yuval Konforti, Nissim Peretz, Xuehan Wang**; the first-class undergraduate students: **Roy Asulin, Yacov Attias, Tamir Bitton, Dror Pezo, Zohar Rimon**; and finally the dedicated postdoctoral research associate: **Dr. Rajib Sharma**, who checked and revised the solutions in the manual.

A partial solution manual (solutions to the first five problems in each chapter) is available on `https://israelcohen.com/publications/books/`. Lecturers who adopt the textbook for teaching can obtain the complete solution manual for all the 250 problems in the textbook by sending a request to Prof. Israel Cohen `icohen@ee.technion.ac.il`.

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2
Single-Channel Signal Enhancement in the Time Domain

Problems

2.1. *Show that the MSEs,* $J(\mathbf{h})$, $J_{\mathrm{d}}(\mathbf{h})$, *and* $J_{\mathrm{n}}(\mathbf{h})$, *are related to the different performance measures by*

$$J(\mathbf{h}) = \sigma_v^2 \left[\mathrm{iSNR} \times \upsilon_{\mathrm{d}}(\mathbf{h}) + \frac{1}{\xi_{\mathrm{n}}(\mathbf{h})} \right],$$

and

$$\begin{aligned} \frac{J_{\mathrm{d}}(\mathbf{h})}{J_{\mathrm{n}}(\mathbf{h})} &= \mathrm{iSNR} \times \xi_{\mathrm{n}}(\mathbf{h}) \times \upsilon_{\mathrm{d}}(\mathbf{h}) \\ &= \mathrm{oSNR}(\mathbf{h}) \times \xi_{\mathrm{d}}(\mathbf{h}) \times \upsilon_{\mathrm{d}}(\mathbf{h}). \end{aligned}$$

Solution Recall that

- $$\mathrm{iSNR} \triangleq \frac{\mathrm{tr}(\mathbf{R}_{\mathbf{x}})}{\mathrm{tr}(\mathbf{R}_{\mathbf{v}})} = \frac{\sigma_x^2}{\sigma_v^2}.$$

- $$\upsilon_{\mathrm{d}}(\mathbf{h}) \triangleq \frac{E\left[\left(x_{\mathrm{fd}}(t) - x(t) \right)^2 \right]}{E\left[x^2(t) \right]} = \frac{(\mathbf{h} - \mathbf{i}_{\mathrm{i}})^T \mathbf{R}_{\mathbf{x}} (\mathbf{h} - \mathbf{i}_{\mathrm{i}})}{\sigma_x^2}.$$

- $$\xi_{\mathrm{n}}(\mathbf{h}) \triangleq \frac{\sigma_v^2}{\mathbf{h}^T \mathbf{R}_v \mathbf{h}}.$$

- $$J(\mathbf{h}) = E\left[e^2(t) \right] = E\left[\left((\mathbf{h} - \mathbf{i}_{\mathrm{i}}) \mathbf{x}(t) + \mathbf{h}^T \mathbf{v}(t) \right)^2 \right].$$

Starting from the last bullet, we have,

$$\begin{aligned} J(\mathbf{h}) &= \mathbb{E}\left[\left((\mathbf{h}-\mathbf{i}_{\mathrm{i}})^T \mathbf{x}(t) + \mathbf{h}^T \mathbf{v}(t)\right)^2\right] \\ &= (\mathbf{h}-\mathbf{i}_{\mathrm{i}})^T \mathbf{R}_{\mathbf{x}} (\mathbf{h}-\mathbf{i}_{\mathrm{i}}) + \mathbf{h}^T \mathbf{R}_{\mathbf{v}} \mathbf{h} \\ &= \sigma_x^2 v_{\mathrm{d}}(\mathbf{h}) + \frac{\sigma_v^2}{\xi_{\mathrm{n}}(\mathbf{h})} \\ &= \sigma_v^2 \left(\mathrm{iSNR} \cdot \nu_{\mathrm{d}}(\mathbf{h}) + \frac{1}{\xi_{\mathrm{n}}(\mathbf{h})}\right). \end{aligned}$$

Again, recall that

- $$J_{\mathrm{d}}(\mathbf{h}) = (\mathbf{h}-\mathbf{i}_{\mathrm{i}})^T \mathbf{R}_{\mathbf{x}} (\mathbf{h}-\mathbf{i}_{\mathrm{i}}) = \sigma_x^2 v_{\mathrm{d}}(\mathbf{h}),$$

- $$J_{\mathrm{n}}(\mathbf{h}) = \mathbf{h}^T \mathbf{R}_{\mathbf{v}} \mathbf{h} = \frac{\sigma_v^2}{\xi_{\mathrm{n}}(\mathbf{h})}.$$

Thus,

$$\frac{J_{\mathrm{d}}(\mathbf{h})}{J_{\mathrm{n}}(\mathbf{h})} = \frac{\sigma_x^2 v_{\mathrm{d}}(\mathbf{h})}{\frac{\sigma_v^2}{\xi_{\mathrm{n}}(\mathbf{h})}} = \mathrm{iSNR} \cdot \xi_{\mathrm{n}}(\mathbf{h}) \cdot \nu_{\mathrm{d}}(\mathbf{h}).$$

■

2.2. *Show that taking the gradient of the MSE :*

$$J(\mathbf{h}) = \sigma_x^2 - 2\mathbf{h}^T \mathbf{R}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}} + \mathbf{h}^T \mathbf{R}_{\mathbf{y}} \mathbf{h},$$

with respect to $\mathbf{h}$ *and equating the result to zero yields the Wiener filter:*

$$\mathbf{h}_{\mathrm{W}} = \mathbf{R}_{\mathbf{y}}^{-1} \mathbf{R}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}.$$

Solution

$$\begin{aligned} \nabla_{\mathbf{h}} J(\mathbf{h}_{\mathrm{W}}) &= \mathbf{0} \\ -2\mathbf{R}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}} + 2\mathbf{R}_{\mathbf{y}} \mathbf{h}_{\mathrm{W}} &= \mathbf{0} \\ \mathbf{h}_{\mathrm{W}} &= \mathbf{R}_{\mathbf{y}}^{-1} \mathbf{R}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}. \end{aligned}$$

■

2.3. *Show that the Wiener filter can be expressed as*

$$\mathbf{h}_{\mathrm{W}} = \left[\mathbf{I}_L - \rho^2(v, y) \boldsymbol{\Gamma}_{\mathbf{y}}^{-1} \boldsymbol{\Gamma}_{\mathbf{v}}\right] \mathbf{i}_{\mathrm{i}}.$$

Solution Recall that

$$\mathbf{R}_{\mathbf{y}} = \mathbf{R}_{\mathbf{x}} + \mathbf{R}_{\mathbf{v}} .$$

Hence,

$$\begin{aligned}
\mathbf{h}_{\mathrm{W}} &= \mathbf{R}_{\mathbf{y}}^{-1}\mathbf{R}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}} \\
&= \mathbf{R}_{\mathbf{y}}^{-1}\left(\mathbf{R}_{\mathbf{y}} - \mathbf{R}_{\mathbf{v}}\right)\mathbf{i}_{\mathrm{i}} \\
&= \left(\mathbf{I}_L - \mathbf{R}_{\mathbf{y}}^{-1}\mathbf{R}_{\mathbf{v}}\right)\mathbf{i}_{\mathrm{i}} \\
&= \left(\mathbf{I}_L - \frac{\sigma_v^2}{\sigma_y^2}\boldsymbol{\Gamma}_{\mathbf{y}}^{-1}\boldsymbol{\Gamma}_{\mathbf{v}}\right)\mathbf{i}_{\mathrm{i}} \\
&= \left(\mathbf{I}_L - \rho^2\left(v, y\right)\boldsymbol{\Gamma}_{\mathbf{y}}^{-1}\boldsymbol{\Gamma}_{\mathbf{v}}\right)\mathbf{i}_{\mathrm{i}} .
\end{aligned}$$

■

2.4. *Prove that* $z_{\mathrm{W}}(t)$*, the estimate of the desired signal with the Wiener filter, satisfies*

$$\rho^2\left(x, z_{\mathrm{W}}\right) \leq \frac{\mathrm{oSNR}\left(\mathbf{h}_{\mathrm{W}}\right)}{1 + \mathrm{oSNR}\left(\mathbf{h}_{\mathrm{W}}\right)}.$$

Solution Note that

$$\begin{aligned}
\rho^2\left(x, z\right) &= \rho^2\left(x, \mathbf{h}^T\mathbf{y}\right) \\
&= \rho^2\left(\mathbf{i}_{\mathrm{i}}^T\mathbf{x}, \mathbf{h}^T\mathbf{y}\right) \\
&= \frac{\mathbb{E}^2\left[\mathbf{i}_{\mathrm{i}}^T\mathbf{x} \cdot \mathbf{h}^T\mathbf{y}\right]}{\sigma_x^2 \cdot \mathbf{h}^T\mathbf{R}_{\mathbf{y}}\mathbf{h}} \\
&= \frac{\left(\mathbf{i}_{\mathrm{i}}^T\mathbf{R}_x\mathbf{h}\right)^2}{\sigma_x^2 \cdot \mathbf{h}^T\mathbf{R}_{\mathbf{y}}\mathbf{h}} ,
\end{aligned}$$

$$\rho^2\left(x, x_{\mathrm{fd}}\right) = \rho^2\left(x, \mathbf{h}^T\mathbf{x}\right) = \frac{\left(\mathbf{i}_{\mathrm{i}}\mathbf{R}_x\mathbf{h}\right)^2}{\sigma_x^2 \cdot \mathbf{h}^T\mathbf{R}_{\mathbf{x}}\mathbf{h}} ,$$

$$\rho^2\left(x_{\mathrm{fd}}, z\right) = \rho^2\left(\mathbf{h}^T\mathbf{x}, \mathbf{h}^T\mathbf{y}\right) = \frac{\left(\mathbf{h}^T\mathbf{R}_{\mathbf{x}}\mathbf{h}\right)^2}{\mathbf{h}^T\mathbf{R}_{\mathbf{x}}\mathbf{h}\left(\mathbf{h}^T\mathbf{R}_{\mathbf{y}}\mathbf{h}\right)} .$$

Thus,

$$\begin{aligned}
\rho^2\left(x, x_{\mathrm{fd}}\right) \cdot \rho^2\left(x_{\mathrm{fd}}, z\right) &= \frac{\left(\mathbf{i}_{\mathrm{i}}\mathbf{R}_x\mathbf{h}\right)^2}{\sigma_x^2 \cdot \mathbf{h}^T\mathbf{R}_{\mathbf{x}}\mathbf{h}} \cdot \frac{\left(\mathbf{h}^T\mathbf{R}_{\mathbf{x}}\mathbf{h}\right)^2}{\mathbf{h}^T\mathbf{R}_{\mathbf{x}}\mathbf{h}\left(\mathbf{h}^T\mathbf{R}_{\mathbf{y}}\mathbf{h}\right)} \\
&= \frac{\left(\mathbf{i}_{\mathrm{i}}\mathbf{R}_x\mathbf{h}\right)^2}{\sigma_x^2\mathbf{h}^T\mathbf{R}_{\mathbf{y}}\mathbf{h}} \\
&= \rho^2\left(x, z\right) .
\end{aligned}$$

Since $\rho^2 \leq 1$ (Cauchy–Schwarz inequality), we have,

$$\rho^2(x, z) \leq \rho^2(x_{\mathrm{fd}}, z) \ .$$

Now,

$$\begin{aligned}
\rho^2(x_{\mathrm{fd}}, z) &= \frac{\left(\mathbf{h}^T \mathbf{R}_{\mathbf{x}} \mathbf{h}\right)^2}{\mathbf{h}^T \mathbf{R}_{\mathbf{x}} \mathbf{h}\left(\mathbf{h}^T \mathbf{R}_{\mathbf{y}} \mathbf{h}\right)} \\
&= \frac{\mathbf{h}^T \mathbf{R}_{\mathbf{x}} \mathbf{h}}{\mathbf{h}^T \left(\mathbf{R}_{\mathbf{x}} + \mathbf{R}_{\mathbf{v}}\right) \mathbf{h}} \\
&= \frac{\sigma^2_{x_{\mathrm{fd}}}}{\sigma^2_{x_{\mathrm{fd}}} + \sigma^2_{v_{\mathrm{rn}}}} \\
&= \frac{\mathrm{oSNR}(\mathbf{h})}{\mathrm{oSNR}(\mathbf{h}) + 1}.
\end{aligned}$$

Hence,

$$\rho^2(x, z) \leq \frac{\mathrm{oSNR}(\mathbf{h})}{\mathrm{oSNR}(\mathbf{h}) + 1},$$

and specifically, using the Wiener filter, we have,

$$\rho^2(x, z_{\mathrm{W}}) \leq \frac{\mathrm{oSNR}(\mathbf{h}_{\mathrm{W}})}{\mathrm{oSNR}(\mathbf{h}_{\mathrm{W}}) + 1} \ .$$

■

2.5. *Prove that with the optimal Wiener filter, the output SNR is always greater than or equal to the input SNR, i.e.,* $\mathrm{oSNR}(\mathbf{h}_{\mathrm{W}}) \geq \mathrm{iSNR}$.

Solution Note that

$$\rho^2(x,y) = \frac{\sigma_x^4}{\sigma_x^2 \sigma_y^2} = \frac{\sigma_x^2}{\sigma_y^2} = \frac{\text{iSNR}}{1+\text{iSNR}},$$

$$\mathbf{h}_{\text{W}} = \mathbf{R}_{\mathbf{y}}^{-1}\mathbf{R}_{\mathbf{x}}\mathbf{i}_{\text{i}},$$

$$\begin{aligned}\rho^2(x, z_{\text{W}}) &= \frac{\left(\mathbf{i}_{\text{i}}^T \mathbf{R}_{\mathbf{x}} \mathbf{h}_{\text{W}}\right)^2}{\sigma_x^2 \cdot \mathbf{h}_{\text{W}}^T \mathbf{R}_{\mathbf{y}} \mathbf{h}_{\text{W}}} \\ &= \frac{\left(\mathbf{i}_{\text{i}}^T \mathbf{R}_{\mathbf{x}} \mathbf{h}_{\text{W}}\right)^2}{\sigma_x^2 \cdot \mathbf{h}_{\text{W}}^T \mathbf{R}_{\mathbf{x}} \mathbf{i}_{\text{i}}} \\ &= \frac{\mathbf{i}_{\text{i}}^T \mathbf{R}_{\mathbf{x}} \mathbf{h}_{\text{W}}}{\sigma_x^2},\end{aligned}$$

$$\begin{aligned}\rho^2(y, z_{\text{W}}) &= \rho^2\left(y, \mathbf{h}_{\text{W}}^T \mathbf{y}\right) \\ &= \frac{\left(\mathbf{i}_{\text{i}}^T \mathbf{R}_{\mathbf{y}} \mathbf{h}_{\text{W}}\right)^2}{\sigma_y^2\left(\mathbf{h}_{\text{W}}^T \mathbf{R}_{\mathbf{y}} \mathbf{h}_{\text{W}}\right)} \\ &= \frac{\left(\mathbf{i}_{\text{i}}^T \mathbf{R}_{\mathbf{x}} \mathbf{i}_{\text{i}}\right)^2}{\sigma_y^2 \mathbf{h}_{\text{W}}^T \mathbf{R}_{\mathbf{x}} \mathbf{i}_{\text{i}}} \\ &= \frac{\sigma_x^4}{\sigma_y^2 \mathbf{h}_{\text{W}}^T \mathbf{R}_{\mathbf{x}} \mathbf{i}_{\text{i}}}.\end{aligned}$$

Hence,

$$\begin{aligned}\rho^2(x,y) = \rho^2(x, z_{\text{W}}) \cdot \rho^2(y, z_{\text{W}}) &\le \rho^2(x, z_{\text{W}}) \\ \frac{\text{iSNR}}{1+\text{iSNR}} &\le \frac{\text{oSNR}(\mathbf{h}_{\text{W}})}{1+\text{oSNR}(\mathbf{h}_{\text{W}})} \\ \text{oSNR}(\mathbf{h}_{\text{W}}) &\ge \text{iSNR}.\end{aligned}$$

■

2.6. *Show that the MMSE can be expressed as*

$$\begin{aligned}J(\mathbf{h}_{\text{W}}) &= \sigma_x^2\left[1 - \rho^2(x, z_{\text{W}})\right] \\ &= \sigma_v^2\left[1 - \rho^2(v, y - z_{\text{W}})\right].\end{aligned}$$

3
Single-Channel Signal Enhancement in the Frequency Domain

Problems

3.1. *Show that the narrowband MSE is given by*

$$J\left[H(f)\right] = \left|1 - H(f)\right|^2 \phi_X(f) + \left|H(f)\right|^2 \phi_V(f).$$

Solution

$$\begin{aligned}
J\left[H\left(f\right)\right] &= \mathbb{E}\left[\left|\mathcal{E}\left(f\right)\right|^2\right] \\
&= \mathbb{E}\left[\left|Z\left(f\right) - X\left(f\right)\right|^2\right] \\
&= \mathbb{E}\left[\left|H\left(f\right)Y\left(f\right) - X\left(f\right)\right|^2\right] \\
&= \mathbb{E}\left[\left|H\left(f\right)\left(X\left(f\right) + V\left(f\right)\right) - X\left(f\right)\right|^2\right] \\
&= \mathbb{E}\left[\left|H\left(f\right)\left(X\left(f\right) + V\left(f\right)\right) - X\left(f\right)\right|^2\right] \\
&= \mathbb{E}\left[\left|\left(H\left(f\right) - 1\right)X\left(f\right) + H\left(f\right)V\left(f\right)\right|^2\right] \\
&= \mathbb{E}\left[\left|\left(H\left(f\right) - 1\right)X\left(f\right)\right|^2\right] + \mathbb{E}\left[\left|H\left(f\right)V\left(f\right)\right|^2\right] \\
&= \left|1 - H\left(f\right)\right|^2 \mathbb{E}\left[\left|X\left(f\right)\right|^2\right] + \left|H\left(f\right)\right|^2 \mathbb{E}\left[\left|V\left(f\right)\right|^2\right] \\
&= \left|1 - H\left(f\right)\right|^2 \phi_X\left(f\right) + \left|H\left(f\right)\right|^2 \phi_V\left(f\right).
\end{aligned}$$

■

3.2. *Show that the narrowband MSE is related to the different narrowband performance measures by*

$$J\left[H(f)\right] = \phi_V(f)\left\{\mathrm{iSNR}(f) \times \upsilon_{\mathrm{d}}\left[H(f)\right] + \frac{1}{\xi_{\mathrm{n}}\left[H(f)\right]}\right\}.$$

Solution Using the previous section,

$$\begin{aligned} J\left[H\left(f\right)\right] &= \left|1 - H\left(f\right)\right|^2 \phi_X\left(f\right) + \left|H\left(f\right)\right|^2 \phi_V\left(f\right) \\ &= \phi_V\left(f\right)\left(\left|1 - H\left(f\right)\right|^2 \frac{\phi_X\left(f\right)}{\phi_V\left(f\right)} + \left|H\left(f\right)\right|^2\right) \\ &= \phi_V\left(f\right)\left(\upsilon_{\mathrm{d}}\left[H\left(f\right)\right] \times \mathrm{iSNR}\left(f\right) + \frac{1}{\xi_{\mathrm{n}}\left[H\left(f\right)\right]}\right). \end{aligned}$$

■

3.3. *Show that the narrowband MSEs* $J_{\mathrm{d}}\left[H(f)\right]$ *and* $J_{\mathrm{n}}\left[H(f)\right]$ *are related to the different narrowband performance measures by*

$$\begin{aligned} \frac{J_{\mathrm{d}}\left[H(f)\right]}{J_{\mathrm{n}}\left[H(f)\right]} &= \mathrm{iSNR}(f) \times \xi_{\mathrm{n}}\left[H(f)\right] \times \upsilon_{\mathrm{d}}\left[H(f)\right] \\ &= \mathrm{oSNR}\left[H(f)\right] \times \xi_{\mathrm{d}}\left[H(f)\right] \times \upsilon_{\mathrm{d}}\left[H(f)\right]. \end{aligned}$$

Solution

$$\begin{aligned} \frac{J_{\mathrm{d}}\left[H\left(f\right)\right]}{J_{\mathrm{n}}\left[H\left(f\right)\right]} &= \frac{\phi_X\left(f\right)\upsilon_{\mathrm{d}}\left[H\left(f\right)\right]}{\frac{\phi_V\left(f\right)}{\xi_n\left[H\left(f\right)\right]}} \\ &= \frac{\phi_X\left(f\right)}{\phi_V\left(f\right)}\xi_n\left[H\left(f\right)\right]\upsilon_{\mathrm{d}}\left[H\left(f\right)\right] \\ &= \mathrm{iSNR}\left(f\right)\xi_n\left[H\left(f\right)\right]\upsilon_{\mathrm{d}}\left[H\left(f\right)\right] \\ &= \mathrm{oSNR}\left(f\right)\xi_{\mathrm{d}}\left[H\left(f\right)\right]\upsilon_{\mathrm{d}}\left[H\left(f\right)\right]. \end{aligned}$$

■

3.4. *Show that the Wiener gain is given by*

$$H_{\mathrm{W}}(f) = \frac{\mathrm{iSNR}(f)}{1 + \mathrm{iSNR}(f)}.$$

Solution Since $\left|1 - H\left(f\right)\right| \geq \left|1 - \left|H\left(f\right)\right|\right|$, we have,

$$\begin{aligned} H_{\mathrm{W}}\left(f\right) &= \arg\min_{H\left(f\right)} J\left[H\left(f\right)\right] \\ &= \arg\min_{H\left(f\right)}\left(\left|1 - H\left(f\right)\right|^2 \phi_X\left(f\right) + \left|H\left(f\right)\right|^2 \phi_V\left(f\right)\right) \\ &= \arg\min_{H\left(f\right)}\left(\left|1 - \left|H\left(f\right)\right|\right|^2 \phi_X\left(f\right) + \left|H\left(f\right)\right|^2 \phi_V\left(f\right)\right) \\ &= \arg\min_{\left|H\left(f\right)\right|}\left(\left(1 - \left|H\left(f\right)\right|\right)^2 \phi_X\left(f\right) + \left|H\left(f\right)\right|^2 \phi_V\left(f\right)\right). \end{aligned}$$

Now,

$$\begin{aligned}\nabla_{|H(f)|} J\left[H\left(f\right)\right] &= 0\\ \nabla_{|H(f)|}\left(\left(1-\left|H\left(f\right)\right|\right)^2 \phi_X\left(f\right) + \left|H\left(f\right)\right|^2 \phi_V\left(f\right)\right) &= 0\\ -2\left(1-H\left(f\right)\right)\phi_X\left(f\right) + 2\left|H\left(f\right)\right|\phi_V\left(f\right) &= 0\\ H\left(f\right)\left(\phi_X\left(f\right)+\phi_V\left(f\right)\right) &= \phi_X\left(f\right)\\ \left|H\left(f\right)\right| &= \frac{\phi_X\left(f\right)}{\phi_X\left(f\right)+\phi_V\left(f\right)}.\end{aligned}$$

Hence,

$$H_{\mathrm{W}}\left(f\right) = \frac{\phi_X\left(f\right)}{\phi_Y\left(f\right)} = \frac{\phi_X\left(f\right)}{\phi_V\left(f\right)+\phi_X\left(f\right)} = \frac{\mathrm{iSNR}\left(f\right)}{1+\mathrm{iSNR}\left(f\right)}.$$

■

3.5. *Show that the Wiener gain is equal to the MSCF between $X(f)$ and $Y(f)$, i.e.,*

$$H_{\mathrm{W}}(f) = \left|\rho\left[X(f), Y(f)\right]\right|^2.$$

Solution

$$\begin{aligned}\left|\rho\left[X\left(f\right), Y\left(f\right)\right]\right|^2 &= \frac{\left|\mathbb{E}\left[X\left(f\right)Y^*\left(f\right)\right]\right|^2}{\mathbb{E}\left[\left|X\left(f\right)\right|^2\right]\mathbb{E}\left[\left|Y\left(f\right)\right|^2\right]}\\ &= \frac{\left|\mathbb{E}\left[X\left(f\right)\left(X^*\left(f\right)+V^*\left(f\right)\right)\right]\right|^2}{\phi_X\left(f\right)\left(\phi_X\left(f\right)+\phi_V\left(f\right)\right)}\\ &= \frac{\left(\phi_X\left(f\right)\right)^2}{\phi_X\left(f\right)\left(\phi_X\left(f\right)+\phi_V\left(f\right)\right)}\\ &= \frac{\phi_X\left(f\right)}{\phi_X\left(f\right)+\phi_V\left(f\right)}\\ &= \frac{\mathrm{iSNR}\left(f\right)}{1+\mathrm{iSNR}\left(f\right)}\\ &= H_{\mathrm{W}}\left(f\right).\end{aligned}$$

■

3.6. *Show that the MMSE can be expressed as*

$$J\left[H_{\mathrm{W}}(f)\right] = \left[1-H_{\mathrm{W}}(f)\right]\phi_X(f).$$

4
Multichannel Signal Enhancement in the Time Domain

Problems

4.1. *Show that if two symmetric matrices* $\mathbf{R}_{\underline{\mathbf{x}}}$ *and* $\mathbf{R}_{\underline{\mathbf{v}}}$ *are jointly diagonalized, i.e.,*

$$\underline{\mathbf{T}}^T \mathbf{R}_{\underline{\mathbf{x}}} \underline{\mathbf{T}} = \underline{\mathbf{\Lambda}},$$
$$\underline{\mathbf{T}}^T \mathbf{R}_{\underline{\mathbf{v}}} \underline{\mathbf{T}} = \mathbf{I}_{ML},$$

then $\underline{\mathbf{\Lambda}}$ *and* $\underline{\mathbf{T}}$ *are, respectively, the eigenvalue and eigenvector matrices of* $\mathbf{R}_{\underline{\mathbf{v}}}^{-1} \mathbf{R}_{\underline{\mathbf{x}}}$*, i.e.,*

$$\mathbf{R}_{\underline{\mathbf{v}}}^{-1} \mathbf{R}_{\underline{\mathbf{x}}} \underline{\mathbf{T}} = \underline{\mathbf{T}}\, \underline{\mathbf{\Lambda}}.$$

Solution

$$\mathbf{R}_{\underline{\mathbf{v}}}^{-1} \mathbf{R}_{\underline{\mathbf{x}}} \underline{\mathbf{T}} = \mathbf{R}_{\underline{\mathbf{v}}}^{-1} \underline{\mathbf{T}}^{-T} \underline{\mathbf{T}}^T \mathbf{R}_{\underline{\mathbf{x}}} \underline{\mathbf{T}} = \underline{\mathbf{T}}\underline{\mathbf{T}}^{-1} \mathbf{R}_{\underline{\mathbf{v}}}^{-1} \underline{\mathbf{T}}^{-T} \underline{\mathbf{\Lambda}} = \underline{\mathbf{T}} \left(\underline{\mathbf{T}}^T \mathbf{R}_{\underline{\mathbf{v}}} \underline{\mathbf{T}} \right)^{-1} \underline{\mathbf{\Lambda}} = \underline{\mathbf{T}}\underline{\mathbf{\Lambda}}.$$

■

4.2. *Denote by* $\underline{\mathbf{t}}_1, \underline{\mathbf{t}}_2, \ldots, \underline{\mathbf{t}}_{ML}$*, the eigenvectors of* $\mathbf{R}_{\underline{\mathbf{v}}}^{-1} \mathbf{R}_{\underline{\mathbf{x}}}$*. Show that*

$$\mathbf{R}_{\underline{\mathbf{v}}}^{-1} = \sum_{i=1}^{ML} \underline{\mathbf{t}}_i \underline{\mathbf{t}}_i^T.$$

Solution

$$\underline{\mathbf{T}}^T \mathbf{R}_{\underline{\mathbf{v}}} \underline{\mathbf{T}} = \mathbf{I}_{ML}$$
$$\mathbf{R}_{\underline{\mathbf{v}}} = \underline{\mathbf{T}}^{-T} \underline{\mathbf{T}}^{-1}$$
$$\mathbf{R}_{\underline{\mathbf{v}}}^{-1} = \underline{\mathbf{T}}\underline{\mathbf{T}}^T = \sum_{i=1}^{ML} \underline{\mathbf{t}}_{\mathrm{i}} \underline{\mathbf{t}}_{\mathrm{i}}^T.$$

■

4.3. *Show that the MSE can be written as*

$$J(\underline{\mathbf{A}}) = \text{Tr}\left[\mathbf{R}_{\mathbf{x}_1} - 2\underline{\mathbf{A}}\,\underline{\mathbf{T}}^T\mathbf{R}_{\underline{\mathbf{x}}}\underline{\mathbf{I}}_{\text{i}}^T + \underline{\mathbf{A}}\left(\underline{\mathbf{\Lambda}} + \mathbf{I}_{ML}\right)\underline{\mathbf{A}}^T\right].$$

Solution

$$\begin{aligned}
J(\underline{\mathbf{A}}) &= \text{Tr}\left\{\mathbb{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right]\right\} \\
&= \text{Tr}\left\{\mathbb{E}\left[\left(\underline{\mathbf{A}\mathbf{T}}^T\underline{\mathbf{y}}(t) - \mathbf{x}_1(t)\right)\left(\underline{\mathbf{A}\mathbf{T}}^T\underline{\mathbf{y}}(t) - \mathbf{x}_1(t)\right)^T\right]\right\} \\
&= \text{Tr}\{\mathbb{E}[\underline{\mathbf{A}\mathbf{T}}^T\underline{\mathbf{y}}(t)\underline{\mathbf{y}}^T(t)\underline{\mathbf{T}\mathbf{A}}^T - \mathbf{x}_1(t)\underline{\mathbf{y}}^T(t)\underline{\mathbf{T}\mathbf{A}}^T \\
&- \underline{\mathbf{A}\mathbf{T}}^T\underline{\mathbf{y}}(t)\mathbf{x}_1^T(t) + \mathbf{x}_1(t)\mathbf{x}_1^T(t)]\} \\
&= \text{Tr}\left\{\underline{\mathbf{A}}\left(\underline{\mathbf{\Lambda}} + \mathbf{I}_{ML}\right)\underline{\mathbf{A}}^T - \underline{\mathbf{I}}_{\text{i}}\mathbf{R}_{\underline{\mathbf{x}}}(t)\underline{\mathbf{T}\mathbf{A}}^T - \underline{\mathbf{A}\mathbf{T}}^T\mathbf{R}_{\underline{\mathbf{x}}}\underline{\mathbf{I}}_{\text{i}}^T + \mathbf{R}_{\mathbf{x}_1}\right\} \\
&= \text{Tr}\left\{\underline{\mathbf{A}}\left(\underline{\mathbf{\Lambda}} + \mathbf{I}_{ML}\right)\underline{\mathbf{A}}^T - 2\underline{\mathbf{A}\mathbf{T}}^T\mathbf{R}_{\underline{\mathbf{x}}}\underline{\mathbf{I}}_{\text{i}}^T + \mathbf{R}_{\mathbf{x}_1}\right\}.
\end{aligned}$$

■

4.4. *Show that the different performance measures are related to the MSEs by*

$$\begin{aligned}
\frac{J_{\text{d}}(\underline{\mathbf{A}})}{J_{\text{n}}(\underline{\mathbf{A}})} &= \text{iSNR} \times \xi_{\text{n}}(\underline{\mathbf{A}}) \times \upsilon_{\text{d}}(\underline{\mathbf{A}}) \\
&= \text{oSNR}(\underline{\mathbf{A}}) \times \xi_{\text{d}}(\underline{\mathbf{A}}) \times \upsilon_{\text{d}}(\underline{\mathbf{A}}).
\end{aligned}$$

Solution

$$\begin{aligned}
J_{\text{d}}(\underline{\mathbf{A}}) &= \text{Tr}\left\{\mathbb{E}\left[\mathbf{e}_{\text{d}}(t)\mathbf{e}_{\text{d}}^T(t)\right]\right\} \\
&= \text{Tr}\left\{\mathbb{E}\left[\left(\left(\underline{\mathbf{A}\mathbf{T}}^T - \underline{\mathbf{I}}_{\text{i}}\right)\underline{\mathbf{x}}(t)\right)\left(\left(\underline{\mathbf{A}\mathbf{T}}^T - \underline{\mathbf{I}}_{\text{i}}\right)\underline{\mathbf{x}}(t)\right)^T\right]\right\} \\
&= \text{Tr}\left\{\left(\underline{\mathbf{A}\mathbf{T}}^T - \underline{\mathbf{I}}_{\text{i}}\right)\mathbf{R}_{\underline{\mathbf{x}}}\left(\underline{\mathbf{A}}^T\underline{\mathbf{T}} - \underline{\mathbf{I}}_{\text{i}}^T\right)\right\} \\
&= \text{Tr}\left\{\underline{\mathbf{A}\mathbf{\Lambda}\mathbf{A}}^T - 2\underline{\mathbf{A}\mathbf{T}}^T\mathbf{R}_{\underline{\mathbf{x}}}\underline{\mathbf{I}}_{\text{i}}^T + \mathbf{R}_{\mathbf{x}_1}\right\} \\
&= \text{Tr}\left\{\mathbf{R}_{\mathbf{x}_1}\right\}\upsilon_{\text{d}}(\underline{\mathbf{A}}).
\end{aligned}$$

$$\begin{aligned}
J_{\text{n}}(\underline{\mathbf{A}}) &= \text{Tr}\left\{\mathbb{E}\left[\mathbf{e}_{\text{n}}(t)\mathbf{e}_{\text{n}}^T(t)\right]\right\} \\
&= \text{Tr}\left\{\mathbb{E}\left[\underline{\mathbf{A}\mathbf{T}}^T\underline{\mathbf{v}}(t)\underline{\mathbf{v}}^T(t)\underline{\mathbf{T}\mathbf{A}}^T\right]\right\} \\
&= \text{Tr}\left\{\underline{\mathbf{A}\mathbf{A}}^T\right\} \\
&= \frac{\text{Tr}\left\{\mathbf{R}_{\mathbf{v}_1}\right\}}{\xi_{\text{n}}(\underline{\mathbf{A}})}.
\end{aligned}$$

Hence,

$$\frac{J_{\mathrm{d}}\left(\underline{\mathbf{A}}\right)}{J_{\mathrm{n}}\left(\underline{\mathbf{A}}\right)} = \frac{\mathrm{Tr}\left\{\mathbf{R}_{\mathbf{x}_1}\right\} \upsilon_{\mathrm{d}}\left(\underline{\mathbf{A}}\right)}{\frac{\mathrm{Tr}\left\{\mathbf{R}_{\mathbf{v}_1}\right\}}{\xi_{\mathrm{n}}\left(\underline{\mathbf{A}}\right)}}$$
$$= \mathrm{iSNR} \times \xi_{\mathrm{n}}\left(\underline{\mathbf{A}}\right) \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{A}}\right)$$
$$= \mathrm{oSNR}\left(\underline{\mathbf{A}}\right) \times \xi_{\mathrm{d}}\left(\underline{\mathbf{A}}\right) \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{A}}\right).$$

■

4.5. *Show that the Wiener filtering matrix can be written as*

$$\underline{\mathbf{H}}_{\mathrm{W}} = \underline{\mathbf{I}}_{\mathrm{i}} \mathbf{R}_{\underline{\mathbf{v}}} \sum_{i=1}^{ML} \frac{\underline{\lambda}_i}{1+\underline{\lambda}_i} \underline{\mathbf{t}}_i \underline{\mathbf{t}}_i^T.$$

Solution

$$\underline{\mathbf{A}}_{\mathrm{W}} = \underline{\mathbf{I}}_{\mathrm{i}} \mathbf{R}_{\underline{\mathbf{x}}} \underline{\mathbf{T}} \left(\underline{\mathbf{\Lambda}} + \mathbf{I}_{ML}\right)^{-1}$$
$$= \underline{\mathbf{I}}_{\mathrm{i}} \underline{\mathbf{T}}^{-T} \underline{\mathbf{T}}^{T} \mathbf{R}_{\underline{\mathbf{x}}} \underline{\mathbf{T}} \left(\underline{\mathbf{\Lambda}} + \mathbf{I}_{ML}\right)^{-1}$$
$$= \underline{\mathbf{I}}_{\mathrm{i}} \underline{\mathbf{T}}^{-T} \underline{\mathbf{\Lambda}} \left(\underline{\mathbf{\Lambda}} + \mathbf{I}_{ML}\right)^{-1}.$$

Hence,

$$\underline{\mathbf{H}}_{\mathrm{W}} = \underline{\mathbf{A}}_{\mathrm{W}} \underline{\mathbf{T}}^{T}$$
$$= \underline{\mathbf{I}}_{\mathrm{i}} \underline{\mathbf{T}}^{-T} \underline{\mathbf{T}}^{-1} \underline{\mathbf{T}} \underline{\mathbf{\Lambda}} \left(\underline{\mathbf{\Lambda}} + \mathbf{I}_{ML}\right)^{-1} \underline{\mathbf{T}}^{T}$$
$$= \underline{\mathbf{I}}_{\mathrm{i}} \mathbf{R}_{\underline{\mathbf{v}}} \underline{\mathbf{T}} \underline{\mathbf{\Lambda}} \left(\underline{\mathbf{\Lambda}} + \mathbf{I}_{ML}\right)^{-1} \underline{\mathbf{T}}^{T}$$
$$= \underline{\mathbf{I}}_{\mathrm{i}} \mathbf{R}_{\underline{\mathbf{v}}} \sum_{i=1}^{ML} \frac{\underline{\lambda}_{\mathrm{i}}}{1+\underline{\lambda}_{\mathrm{i}}} \underline{\mathbf{t}}_{\mathrm{i}} \underline{\mathbf{t}}_{\mathrm{i}}^T.$$

■

4.6. *Prove that with the optimal Wiener filtering matrix, the output SNR is always greater than or equal to the input SNR, i.e.,* $\mathrm{oSNR}\left(\underline{\mathbf{H}}_{\mathrm{W}}\right) \geq \mathrm{iSNR}$.

5
Multichannel Signal Enhancement in the Frequency Domain

Problems

5.1. *Assume that the matrix* $\mathbf{\Phi_v}(f)$ *is nonsingular. Show that*

$$\left|\mathbf{h}^H(f)\mathbf{d}(f)\right|^2 \leq \left[\mathbf{h}^H(f)\mathbf{\Phi_v}(f)\mathbf{h}(f)\right]\left[\mathbf{d}^H(f)\mathbf{\Phi_v}^{-1}(f)\mathbf{d}(f)\right],$$

with equality if and only if $\mathbf{h}(f) \propto \mathbf{\Phi_v}^{-1}(f)\mathbf{d}(f)$.

Solution Remember that inner-product is defined as $< x, y >= y^*x$. Hence,

$$\begin{aligned}
&< \mathbf{\Phi_v}^{-1/2}(f)\mathbf{d}(f), \mathbf{\Phi_v}^{1/2}(f)^H\mathbf{h}(f) > \\
&= \left(\mathbf{\Phi_v}^{1/2}(f)^H\mathbf{h}(f)\right)^H\left(\mathbf{\Phi_v}^{-1/2}(f)\mathbf{d}(f)\right) \\
&= \mathbf{h}^H(f)\mathbf{\Phi_v}^{1/2}(f)\mathbf{\Phi_v}^{-1/2}(f)\mathbf{d}(f) = \mathbf{h}^H(f)\mathbf{d}(f).
\end{aligned}$$

Using Cauchy Schwarz inequality, we get,

$$\begin{aligned}
&\left|< \mathbf{\Phi_v}^{-1/2}(f)\mathbf{d}(f), \mathbf{\Phi_v}^{1/2}(f)^H\mathbf{h}(f) >\right|^2 \\
&= \left|\mathbf{h}^H(f)\mathbf{d}(f)\right|^2 \\
&\leq \\
&< \left(\mathbf{\Phi_v}^{1/2}(f)\right)^H\mathbf{h}(f), \left(\mathbf{\Phi_v}^{1/2}(f)\right)^H\mathbf{h}(f) > \cdot < \left(\mathbf{\Phi_v}^{-1/2}(f)\right)^H\mathbf{d}(f), \left(\mathbf{\Phi_v}^{-1/2}(f)\right)^H\mathbf{d}(f) > \\
&= \left[\mathbf{h}^H(f)\mathbf{\Phi_v}^{1/2}(f)\mathbf{\Phi_v}^{1/2}(f)\mathbf{h}(f)\right]\left[\mathbf{d}^H(f)\mathbf{\Phi_v}^{-1/2}(f)\mathbf{\Phi_v}^{-1/2}(f)\mathbf{d}(f)\right] \\
&= \left[\mathbf{h}^H(f)\mathbf{\Phi_v}(f)\mathbf{h}(f)\right]\left[\mathbf{d}^H(f)\mathbf{\Phi_v}^{-1}(f)\mathbf{d}(f)\right].
\end{aligned}$$

Obviously, we get an equality if and only if $\mathbf{h}(f) \propto \mathbf{\Phi_v}^{-1}(f)\mathbf{d}(f)$ since both identities are identical by definition.

■

5.2. *Show that the narrowband output SNR is upper bounded by*

$$\text{oSNR}\left[\mathbf{h}(f)\right] \leq \phi_{X_1}(f) \times \mathbf{d}^H(f)\mathbf{\Phi_v}^{-1}(f)\mathbf{d}(f), \ \forall\, \mathbf{h}(f).$$

Solution Recall that

$$\text{oSNR}\left[\mathbf{h}(f)\right] = \frac{\phi_{X_1}(f) \times \left|\mathbf{h}^H(f)\mathbf{d}(f)\right|^2}{\mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{v}}(f)\mathbf{h}(f)}.$$

Using the inequality from Problem 1, we deduce an upper bound for the narrowband output SNR.

$$\begin{aligned}\text{oSNR}\left[\mathbf{h}(f)\right] &\leq \frac{\phi_{X_1}(f) \times \left[\mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{v}}(f)\mathbf{h}(f)\right]\left[\mathbf{d}^H(f)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}(f)\right]}{\mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{v}}(f)\mathbf{h}(f)} \\ &= \phi_{X_1}(f) \times \mathbf{d}^H(f)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}(f).\end{aligned}$$

■

5.3. *Show that*

$$\text{oSNR}\left[\mathbf{i}_{\text{i}}(f)\right] \leq \phi_{X_1}(f) \times \mathbf{d}^H(f)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}(f).$$

Solution As we show in Problem 2, the upper bound for the narrowband output SNR is independent of $\mathbf{h}(f)$. Hence,

$$\text{oSNR}\left[\mathbf{i}_{\text{i}}(f)\right] \leq \phi_{X_1}(f) \times \mathbf{d}^H(f)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}(f).$$

■

5.4. *Show that*

$$\phi_{V_1}(f) \times \mathbf{d}^H(f)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}(f) \geq 1.$$

Solution Using the Cauchy-Schwarz Inequality, we get,

$$\left|\mathbf{h}^H(f)\mathbf{d}(f)\right|^2 \leq \left[\mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{v}}(f)\mathbf{h}(f)\right]\left[\mathbf{d}^H(f)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}(f)\right]$$

Applying the identity filter to the above equation, we receive,

$$\begin{aligned}\left|\mathbf{i}_{\text{i}}^H\mathbf{d}(f)\right|^2 &= 1, \\ \mathbf{i}_{\text{i}}^H(f)\mathbf{\Phi}_{\mathbf{v}}(f)\mathbf{i}_{\text{i}}(f) &= \phi_{V_1}(f).\end{aligned}$$

Hence,

$$1 \leq \phi_{V_1}(f) \times \mathbf{d}^H(f)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}(f).$$

■

5.5. *Show that the narrowband desired signal distortion index is given by*

$$\upsilon_{\text{d}}\left[\mathbf{h}(f)\right] = \left|\mathbf{h}^H(f)\gamma_{X_1\mathbf{x}}^*(f) - 1\right|^2.$$

Solution The narrowband desired signal distortion index is defined by

$$\upsilon_{\rm d}\left[\mathbf{h}(f)\right] = \frac{\mathbb{E}\left[\left|X_{\rm fd}(f) - X_1(f)\right|^2\right]}{\phi_{X_1}(f)}, \text{ where } \phi_{X_1}(f) = \mathbb{E}\left[|X_1(f)|^2\right].$$

The filtered desired signal is given by,

$$X_{\rm fd}(f) = X_1(f)\mathbf{h}^H(f)\boldsymbol{\gamma}^*_{X_1\mathbf{x}}(f).$$

Hence, we can rewrite the desired signal distortion index as,

$$\upsilon_{\rm d}\left[\mathbf{h}(f)\right] = \frac{\mathbb{E}\left[\left|\left[\mathbf{h}^H(f)\boldsymbol{\gamma}^*_{X_1\mathbf{x}}(f) - 1\right]X_1(f)\right|^2\right]}{\phi_{X_1}(f)} = |\mathbf{h}^H(f)\boldsymbol{\gamma}^*_{X_1\mathbf{x}}(f) - 1|^2.$$

■

5.6. *Show that the narrowband MSE can be written as*

$$J\left[\mathbf{h}(f)\right] = \phi_{X_1}(f) + \mathbf{h}^H(f)\boldsymbol{\Phi}_{\mathbf{y}}(f)\mathbf{h}(f) - \phi_{X_1}(f)\mathbf{h}^H(f)\boldsymbol{\gamma}^*_{X_1\mathbf{x}}(f) - \phi_{X_1}(f)\boldsymbol{\gamma}^T_{X_1\mathbf{x}}(f)\mathbf{h}(f).$$

Solution Defining the error signal between the estimated and desired signals at frequency, f, as

$$\begin{aligned}\mathcal{E}(f) &= Z(f) - X_1(f)\\ &= X_1(f)\mathbf{h}^H(f)\boldsymbol{\gamma}^*_{X_1\mathbf{x}}(f) + \mathbf{h}^H(f)\mathbf{v}(f) - X_1(f)\\ &= \mathcal{E}_{\rm d}(f) + \mathcal{E}_{\rm n}(f),\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_{\rm d}(f) &= \left[\mathbf{h}^H(f)\boldsymbol{\gamma}^*_{X_1\mathbf{x}}(f) - 1\right]X_1(f)\\ \mathcal{E}_{\rm n}(f) &= \mathbf{h}^H(f)\mathbf{v}(f).\end{aligned}$$

The MSE is then defined as the variance of the error signal, i.e.,

$$J\left[\mathbf{h}(f)\right] = \mathbb{E}\left[\left|\mathcal{E}(f)\right|^2\right] = \mathbb{E}\left[\left|\mathcal{E}_{\rm d}(f)\right|^2\right] + \mathbb{E}\left[\left|\mathcal{E}_{\rm n}(f)\right|^2\right],$$

since $\mathcal{E}_{\rm d}(f)$ and $\mathcal{E}_{\rm n}(f)$ are incoherent. Now,

$$\begin{aligned}\mathbb{E}\left[\left|\mathcal{E}_{\rm d}(f)\right|^2\right] =& \phi_{X_1}(f)\mathbf{h}^H(f)\boldsymbol{\gamma}^*_{X_1\mathbf{x}}(f)\boldsymbol{\gamma}^T_{X_1\mathbf{x}}(f)\mathbf{h}(f) -\\ & \phi_{X_1}(f)\mathbf{h}^H(f)\boldsymbol{\gamma}^*_{X_1\mathbf{x}}(f) - \phi_{X_1}(f)\boldsymbol{\gamma}^T_{X_1\mathbf{x}}(f)\mathbf{h}(f) + \phi_{X_1}(f),\\ \mathbb{E}\left[\left|\mathcal{E}_{\rm n}(f)\right|^2\right] =& \mathbf{h}^H(f)\boldsymbol{\Phi}_{\mathbf{v}}(f)\mathbf{h}(f).\end{aligned}$$

6
An Exhaustive Class of Linear Filters

Problems

6.1. *Show that the Wiener filter can be expressed as*

$$\mathbf{h}_{\mathrm{W}} = \left(\mathbf{I}_M - \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{\Phi}_{\mathbf{in}}\right)\mathbf{i}_{\mathrm{i}}.$$

Solution Recall that Wiener filter minimizes the MSE criteria :

$$J(\mathbf{h}) = \phi_{x_1} + \mathbf{h}^H\mathbf{\Phi}_{\mathbf{y}}\mathbf{h} - \mathbf{h}^H\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}} - \mathbf{i}_{\mathrm{i}}{}^H\mathbf{\Phi}_{\mathbf{x}}\mathbf{h}.$$

Calculating the gradient :

$$\nabla_{\mathbf{h}} J(\mathbf{h}) = \left(\mathbf{\Phi}_{\mathbf{y}} + \mathbf{\Phi}_{\mathbf{y}}{}^H\right)\mathbf{h} - \mathbf{\Phi}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}} - \mathbf{\Phi}_{\mathbf{x}}{}^H\mathbf{i}_{\mathrm{i}}.$$

Recall that $\mathbf{\Phi}_{\mathbf{y}}$ and $\mathbf{\Phi}_{\mathbf{x}}$ are symmetric, we get,

$$2\mathbf{\Phi}_{\mathbf{y}}\mathbf{h} - 2\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}} = \mathbf{0}.$$

Also, $\mathbf{\Phi}_{\mathbf{x}} = \mathbf{\Phi}_{\mathbf{y}} - \mathbf{\Phi}_{\mathbf{in}}$. Therefore,

$$\begin{aligned}
2\mathbf{\Phi}_{\mathbf{y}}\mathbf{h} - 2\left(\mathbf{\Phi}_{\mathbf{y}} - \mathbf{\Phi}_{\mathbf{in}}\right)\mathbf{i}_{\mathrm{i}} &= \mathbf{0} \\
\mathbf{\Phi}_{\mathbf{y}}\mathbf{h} &= \left(\mathbf{\Phi}_{\mathbf{y}} - \mathbf{\Phi}_{\mathbf{in}}\right)\mathbf{i}_{\mathrm{i}} \\
\mathbf{h} &= \mathbf{\Phi}_{\mathbf{y}}{}^{-1}\left(\mathbf{\Phi}_{\mathbf{y}} - \mathbf{\Phi}_{\mathbf{in}}\right)\mathbf{i}_{\mathrm{i}} \\
\mathbf{h}_{\mathrm{W}} &= \left(\mathbf{I}_{\mathbf{M}} - \mathbf{\Phi}_{\mathbf{y}}{}^{-1}\mathbf{\Phi}_{\mathbf{in}}\right)\mathbf{i}_{\mathrm{i}}.
\end{aligned}$$

■

6.2. *Using Woodbury's identity, show that*

$$\mathbf{\Phi}_{\mathbf{y}}^{-1} = \mathbf{\Phi}_{\mathbf{in}}^{-1} - \mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}'_{\mathbf{x}}\left(\mathbf{\Lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^H_{\mathbf{x}}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}'_{\mathbf{x}}\right)^{-1}\mathbf{Q}'^H_{\mathbf{x}}\mathbf{\Phi}_{\mathbf{in}}^{-1}.$$

Solution Recall that $\mathbf{\Phi_y} = \mathbf{\Phi_x} + \mathbf{\Phi_{in}}$, and $\mathbf{\Phi_x}$ are diagonalizable :

$$\boldsymbol{\lambda}'_{\mathbf{x}} = \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi_{in}} \mathbf{Q}'_{\mathbf{x}},$$

where $\mathbf{Q}'_{\mathbf{x}}$ is a unitary matrix and $\boldsymbol{\lambda}'_{\mathbf{x}}$ is a diagonal matrix. Using Woodbury's identity, we get,

$$\begin{aligned}
\mathbf{\Phi_y}^{-1} &= (\mathbf{\Phi_x} + \mathbf{\Phi_{in}})^{-1} \\
&= \left(\mathbf{Q_x} \boldsymbol{\lambda}'_{\mathbf{x}} \mathbf{Q}'^{\mathbf{H}}_{\mathbf{x}} + \mathbf{\Phi_{in}}\right)^{-1} \\
&= \mathbf{\Phi}_{\mathbf{in}}^{-1} - \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\boldsymbol{\lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1}.
\end{aligned}$$

■

6.3. *Show that the Wiener filter can be expressed as*

$$\mathbf{h}_{\mathrm{W}} = \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\boldsymbol{\lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}.$$

Solution The Wiener filter is given by,

$$\mathbf{h}_{\mathrm{W}} = \left(\mathbf{I}_M - \mathbf{\Phi}_{\mathbf{y}}^{-1} \mathbf{\Phi_{in}}\right) \mathbf{i}_{\mathrm{i}}.$$

Also,

$$\mathbf{\Phi}_{\mathbf{y}}^{-1} = \mathbf{\Phi}_{\mathbf{in}}^{-1} - \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\boldsymbol{\lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1}.$$

Hence,

$$\begin{aligned}
\mathbf{h}_{\mathrm{W}} &= \left(\mathbf{I}_M - (\mathbf{\Phi}_{\mathbf{in}}^{-1} - \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\boldsymbol{\lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1}) \mathbf{\Phi_{in}}\right) \mathbf{i}_{\mathrm{i}} \\
&= \left(\mathbf{I}_M - \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{\Phi_{in}} + \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\boldsymbol{\lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{\Phi_{in}}\right) \mathbf{i}_{\mathrm{i}}.
\end{aligned}$$

By employing the relation $\mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{\Phi_{in}} = \mathbf{I}_M$, which is true since $\mathbf{\Phi_{in}}$ is a correlation matrix of rank M, we have,

$$\begin{aligned}
\mathbf{h}_{\mathrm{W}} &= \left(\mathbf{I}_M - \mathbf{I}_M + \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\boldsymbol{\lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{I}_M\right) \mathbf{i}_{\mathrm{i}} \\
&= \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\boldsymbol{\lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}.
\end{aligned}$$

■

6.4. *Show that the MVDR filter is given by*

$$\mathbf{h}_{\mathrm{MVDR}} = \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{in}}^{-1} \mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}.$$

Solution We have,

$$\begin{aligned}\mathbf{h}_{\mathrm{MVDR}} &= \min_{\mathbf{h}} \left[J_{\mathrm{n}}(\mathbf{h}) + J_{\mathrm{i}}(\mathbf{h})\right] \text{ s.t. } \mathbf{h}^H\mathbf{Q}_{\mathbf{x}}^{'} = \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'} \\ &= \min_{\mathbf{h}} \left[\phi_{v_0}\mathbf{h}^H\mathbf{h} + \mathbf{h}^H\mathbf{\Phi}_{\mathbf{v}}\mathbf{h}\right] \text{ s.t. } \mathbf{h}^H\mathbf{Q}_{\mathbf{x}}^{'} = \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'} \\ &= \min_{\mathbf{h}} \left[\mathbf{h}^H\mathbf{\Phi}_{\mathbf{in}}\mathbf{h}\right] \text{ s.t. } \mathbf{h}^H\mathbf{Q}_{\mathbf{x}}^{'} = \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'}.\end{aligned}$$

To obtain this minimal value, let us define the Lagrange multiplier, λ, and construct the following expression to be minimized :

$$J(\mathbf{h}, \lambda) = \mathbf{h}^H\mathbf{\Phi}_{\mathbf{in}}\mathbf{h} + \left(\mathbf{h}^H\mathbf{Q}_{\mathbf{x}}^{'} - \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'}\right)\lambda.$$

Assuming that $\mathbf{h}$, $\mathbf{h}^H$ are independent, we have,

$$\begin{aligned}\nabla_{\mathbf{h}^*(f)} J(\mathbf{h}_{\mathrm{MVDR}}, \lambda) &= \mathbf{0} \\ \mathbf{\Phi}_{\mathbf{in}}\mathbf{h}_{\mathrm{MVDR}} + \mathbf{Q}_{\mathbf{x}}^{'}\lambda &= \mathbf{0} \\ \mathbf{Q}_{\mathbf{x}}^{'}\lambda &= -\mathbf{\Phi}_{\mathbf{in}}\mathbf{h}_{\mathrm{MVDR}}.\end{aligned}$$

Hence,

$$\begin{aligned}\nabla_{\lambda} J(\mathbf{h}_{\mathrm{MVDR}}, \lambda) &= 0 \\ \mathbf{h}_{\mathrm{MVDR}}^H\mathbf{Q}_{\mathbf{x}}^{'} &= \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'}. \\ \mathbf{h}_{\mathrm{MVDR}}^H\mathbf{Q}_{\mathbf{x}}^{'}\lambda &= \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'}\lambda \\ \mathbf{h}_{\mathrm{MVDR}}^H\mathbf{\Phi}_{\mathbf{in}}\mathbf{h}_{\mathrm{MVDR}} &= \mathbf{i}_{\mathrm{i}}^T\mathbf{\Phi}_{\mathbf{in}}\mathbf{h}_{\mathrm{MVDR}}.\end{aligned}$$

Now,

$$\begin{aligned}\left(\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{i}_{\mathrm{i}}\right)^H\mathbf{Q}_{\mathbf{x}}^{'} &= \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'} \\ \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{'} &= \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'} \\ \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{\prime} &= \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{'}.\end{aligned}$$

Now,

$$\begin{aligned}&\left(\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{i}_{\mathrm{i}}\right)^H\mathbf{\Phi}_{\mathbf{in}}\left(\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{i}_{\mathrm{i}}\right) \\ &= \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{\Phi}_{\mathbf{in}}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{i}_{\mathrm{i}} \\ &= \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{i}_{\mathrm{i}}.\end{aligned}$$

Now,

$$\begin{aligned}&\mathbf{i}_{\mathrm{i}}^T\mathbf{\Phi}_{\mathbf{in}}\left(\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{i}_{\mathrm{i}}\right) \\ &= \mathbf{i}_{\mathrm{i}}^T\mathbf{Q}_{\mathbf{x}}^{\prime}\left(\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime}\right)^{-1}\mathbf{Q}_{\mathbf{x}}^{\prime H}\mathbf{i}_{\mathrm{i}}.\end{aligned}$$

■

6.5. *Show that the MVDR filter can be expressed as*

$$\mathbf{h}_{\mathrm{MVDR}} = \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}.$$

Solution The solution of MVDR optimization problem is :

$$\mathbf{h}_{\mathrm{MVDR}} = \mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}.$$

Recall that

$$\mathbf{\Phi}_{\mathbf{in}} = \mathbf{\Phi}_{\mathbf{y}} - \mathbf{\Phi}_{\mathbf{x}} = \mathbf{\Phi}_{\mathbf{y}} - \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' = \mathbf{\Phi}_{\mathbf{y}} - \mathbf{Q}_{\mathbf{x}}'\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'^{H},$$

Where $\mathbf{Q}_{\mathbf{x}}'$ is unitary. Using Woodbury's identity, the inverse of $\Phi_{\mathbf{in}}$ is :

$$\begin{aligned}
\mathbf{\Phi}_{\mathbf{in}}^{-1} &= \left(\mathbf{\Phi}_{\mathbf{y}} - \mathbf{Q}_{\mathbf{x}}'\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'^{H}\right)^{-1} \\
&= \mathbf{\Phi}_{\mathbf{y}}^{-1} + \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\boldsymbol{\lambda}_{\mathbf{x}}'^{-1} - \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{y}}^{-1} \\
&= \mathbf{\Phi}_{\mathbf{y}}^{-1} \left(\mathbf{I}_{\mathrm{M}} - \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\boldsymbol{\lambda}_{\mathbf{x}}'^{-1} - \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{y}}\right)^{-1} \\
&= \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{A}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{h}_{\mathrm{MVDR}} &= \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{A}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{A}\mathbf{Q}_{\mathbf{x}}'\right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}} \\
&= \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{A}\mathbf{Q}_{\mathbf{x}}'\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{A}^{-1}\mathbf{\Phi}_{\mathbf{y}}\mathbf{Q}_{\mathbf{x}}'\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}} \\
&= \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{\Phi}_{\mathbf{y}}\mathbf{i}_{\mathrm{i}} \\
&= \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}'\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{y}}\mathbf{Q}_{\mathbf{x}}'\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}} \\
&= \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}.
\end{aligned}$$

■

6.6. *Show that the tradeoff filter is given by*

$$\begin{aligned}
\mathbf{h}_{\mathrm{T},\mu} &= \left(\mathbf{\Phi}_{\mathbf{x}} + \mu\mathbf{\Phi}_{\mathbf{in}}\right)^{-1}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}} \\
&= \left[\mathbf{\Phi}_{\mathbf{y}} + (\mu - 1)\mathbf{\Phi}_{\mathbf{in}}\right]^{-1}\left(\mathbf{\Phi}_{\mathbf{y}} - \mathbf{\Phi}_{\mathbf{in}}\right)\mathbf{i}_{\mathrm{i}},
\end{aligned}$$

where μ is a Lagrange multiplier.

7
Fixed Beamforming

Problems

7.1. *Show that the WNG can be written as*

$$\mathcal{W}\left[\mathbf{h}(f)\right] = \mathcal{W}_{\max} \cos^2\left[\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right), \mathbf{h}(f)\right].$$

Solution We have,

$$\begin{aligned}
\mathcal{W}\left[\mathbf{h}(f)\right] =& \frac{1}{\mathbf{h}^H\left(f\right)\mathbf{h}\left(f\right)} \\
&= \frac{1}{\left|\left|\mathbf{h}\left(f\right)\right|\right|_2^2} \\
&= \frac{4\left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2^2}{4\left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2^2\left|\left|\mathbf{h}\left(f\right)\right|\right|_2^2} \\
&= \frac{2^2\left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2^2}{4\left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2^2\left|\left|\mathbf{h}\left(f\right)\right|\right|_2^2}.
\end{aligned}$$

We now use the fact that $\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)^H\mathbf{h} = 1 = 1^H = \mathbf{h}^H\left(f\right)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)$.

$$\begin{aligned}
\mathcal{W}\left[\mathbf{h}(f)\right] &= \left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2^2\left[\frac{\mathbf{h}^H\left(f\right)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) + \mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{h}}{2\left|\left|\mathbf{h}\left(f\right)\right|\right|_2\left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2}\right]^2 \\
&= \left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2^2\cos^2\left[\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right), \mathbf{h}(f)\right].
\end{aligned}$$

We calculate $\left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2^2$:

$$\begin{aligned}
\left|\left|\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\right|\right|_2^2 &= \mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) \\
&= \sum_{i=0}^{M-1} e^{-2\pi f\tau_0\cos\theta_{\mathrm{d}} i} e^{2\pi f\tau_0\cos\theta_{\mathrm{d}} i} \\
&= \sum_{i=0}^{M-1} 1 = M.
\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{W}\left[\mathbf{h}(f)\right] &= M^2 \cos^2 \left[\mathbf{d}\left(f, \cos\theta_\mathrm{d}\right), \mathbf{h}(f)\right] \\ &= \mathcal{W}_{\max} \cos^2 \left[\mathbf{d}\left(f, \cos\theta_\mathrm{d}\right), \mathbf{h}(f)\right].\end{aligned}$$

■

7.2. *Using the Cauchy-Schwarz inequality, show that the maximum DF is*

$$\begin{aligned}\mathcal{D}_{\max}\left(f, \cos\theta_\mathrm{d}\right) &= \mathbf{d}^H\left(f, \cos\theta_\mathrm{d}\right) \mathbf{\Gamma}_{0,\pi}^{-1}(f) \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right) \\ &= \mathrm{tr}\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f) \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right) \mathbf{d}^H\left(f, \cos\theta_\mathrm{d}\right)\right] \\ &\leq M \mathrm{tr}\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\right].\end{aligned}$$

Solution First, we will prove that $\mathbf{\Gamma}_{0,\pi}(f)$ is a positive-definite Hermitian matrix. It is well-established that $\mathbf{\Gamma}_{0,\pi}(f)$ is Hermitian. Now,

$$\begin{aligned}\mathbf{y}^H \mathbf{\Gamma}_{0,\pi}(f) \mathbf{y} &= \frac{1}{2} \int_0^\pi \mathbf{y}^H \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right) \mathbf{d}^H\left(f, \cos\theta_\mathrm{d}\right) \mathbf{y} \sin\theta d\theta \\ &= \frac{1}{2} \int_0^\pi \left|\left|\mathbf{y}^H \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right)\right|\right|^2 \sin\theta d\theta.\end{aligned}$$

Notice that this expression is greater than zero, since $\left|\left|\mathbf{y}^H \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right)\right|\right|^2$ and $\sin\theta$ are by definition non-negative for these θ values, and for $\theta = \frac{\pi}{2}$, $\left|\left|\mathbf{y}^H \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right)\right|\right|^2 > 0$ and $\sin\theta > 0$. So we proved that $\mathbf{\Gamma}_{0,\pi}(f)$ is a Hermitian positive-definite matrix. Now, since $\mathbf{\Gamma}_{0,\pi}(f)$ is a Hermitian positive-definite matrix, we can say that there exists a square root to the matrix, marked as $\mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f)$. Using the Cauchy-Schwarz inequality, we have,

$$\begin{aligned}&\left|\mathbf{h}^H(f) \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right)\right|^2 \\ &= \left|\mathbf{h}^H(f) \mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f) \mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f) \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right)\right|^2 \\ &\leq \mathbf{h}^H(f) \mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f) \mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}H}(f) \mathbf{h}(f) \mathbf{d}^H\left(f, \cos\theta_\mathrm{d}\right) \mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}H}(f) \mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f) \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right).\end{aligned}$$

This inequality is equal when

$$\mathbf{h}^H(f) \mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f) = \mathbf{d}^H\left(f, \cos\theta_\mathrm{d}\right) \mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}H}(f),$$

which is possible since $\mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f)$ is the square root of a positive-definite matrix, and as such is invertible. Thus,

$$\begin{aligned}&\mathcal{D}_{\max}\left(f, \cos\theta_\mathrm{d}\right) \\ &= \frac{\mathbf{h}^H(f) \mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f) \mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}H}(f) \mathbf{h}(f) \mathbf{d}^H\left(f, \cos\theta_\mathrm{d}\right) \mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}H}(f) \mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f) \mathbf{d}\left(f, \cos\theta_\mathrm{d}\right)}{\mathbf{h}^H(f) \mathbf{\Gamma}_{0,\pi}(f) \mathbf{h}(f)}.\end{aligned}$$

Notice that $\mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f)$ and $\mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f)$ are also Hermitian matrices. Therefore,

$$\begin{aligned}\mathcal{D}_{\max}\left(f,\cos\theta_{\mathrm{d}}\right) &= \frac{\mathbf{h}^H(f)\mathbf{\Gamma}_{0,\pi}(f)\mathbf{h}(f)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)}{\mathbf{h}^H(f)\mathbf{\Gamma}_{0,\pi}(f)\mathbf{h}(f)} \\ &= \mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right).\end{aligned}$$

Now we will show this explicitly, when indexing from 0 to $M-1$ for the sake of convenience.

$$\begin{aligned}\mathcal{D}_{\max}\left(f,\cos\theta_{\mathrm{d}}\right) &= \sum_{i=0}^{M-1}\sum_{k=0}^{M-1} e^{j2\pi f\tau_0\cos\theta_{\mathrm{d}} i}e^{-j2\pi f\tau_0\cos\theta_{\mathrm{d}} k}\mathbf{\Gamma}_{0,\pi}^{-1}(f)_{i,k} \\ &= \sum_{i=0}^{M-1}\sum_{k=0}^{M-1} e^{j2\pi f\tau_0\cos\theta_{\mathrm{d}}(i-k)}\mathbf{\Gamma}_{0,\pi}^{-1}(f)_{i,k}.\end{aligned}$$

On the other hand, if we express $\mathrm{tr}\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\right]$ explicitly, we get,

$$\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\right]_i = \sum_{k=0}^{M-1}\mathbf{\Gamma}_{0,\pi}^{-1}(f)_{i,k}\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)_k = \sum_{l=0}^{M-1}\mathbf{\Gamma}_{0,\pi}^{-1}(f)_{i,l}e^{-j2\pi f\tau_0\cos\theta_{\mathrm{d}} l},$$

$$\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\right]_{i,k} = \sum_{l=0}^{M-1}\mathbf{\Gamma}_{0,\pi}^{-1}(f)_{i,l}e^{-j2\pi f\tau_0\cos\theta_{\mathrm{d}} l}e^{j2\pi f\tau_0\cos\theta_{\mathrm{d}} k},$$

$$\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\right]_{i,i} = \sum_{l=0}^{M-1}\mathbf{\Gamma}_{0,\pi}^{-1}(f)_{i,l}e^{-j2\pi f\tau_0\cos\theta_{\mathrm{d}} l}e^{j2\pi f\tau_0\cos\theta_{\mathrm{d}} i},$$

$$\mathrm{tr}\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\right] = \sum_{i=0}^{M-1}\sum_{l=0}^{M-1}\mathbf{\Gamma}_{0,\pi}^{-1}(f)_{i,l}e^{j2\pi f\tau_0\cos\theta_{\mathrm{d}}(i-l)}.$$

But this equals to the same expression as $\mathcal{D}_{\max}\left(f,\cos\theta_{\mathrm{d}}\right)$. Therefore,

$$\mathcal{D}_{\max}\left(f,\cos\theta_{\mathrm{d}}\right) = \mathrm{tr}\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\right].$$

Notice that similar to $\mathbf{\Gamma}_{0,\pi}(f)$, $\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)$ is a positive semi-definite matrix. Since $\mathbf{\Gamma}_{0,\pi}(f)$ and $\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)$ are both Hermitian positive semi-definite matrices, we can say that

$$\begin{aligned}
&\mathrm{tr}\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^{H}\left(f,\cos\theta_{\mathrm{d}}\right)\right] \\
&\leq \sum_{i=1}^{M}\lambda_{i,\mathbf{\Gamma}_{0,\pi}^{-1}(f)}\lambda_{i,\mathbf{d}(f,\cos\theta_{\mathrm{d}})\mathbf{d}^{H}(f,\cos\theta_{\mathrm{d}})} \\
&\leq \sum_{i=1}^{M}\lambda_{i,\mathbf{\Gamma}_{0,\pi}^{-1}(f)}\sum_{j=1}^{M}\lambda_{j,\mathbf{d}(f,\cos\theta_{\mathrm{d}})\mathbf{d}^{H}(f,\cos\theta_{\mathrm{d}})} \\
&\quad = \sum_{i=1}^{M}\lambda_{i,\mathbf{\Gamma}_{0,\pi}^{-1}(f)}\mathrm{tr}\left[\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^{H}\left(f,\cos\theta_{\mathrm{d}}\right)\right] \\
&\quad = M\sum_{i=1}^{M}\lambda_{i,\mathbf{\Gamma}_{0,\pi}^{-1}(f)} = M\mathrm{tr}\left[\mathbf{\Gamma}_{0,\pi}^{-1}(f)\right].
\end{aligned}$$

Here we used the fact that all the eigenvalues of a positive semi-definite matrix are non-negative, and

$$\mathrm{tr}\left[\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}^{H}\left(f,\cos\theta_{\mathrm{d}}\right)\right] = M = \mathbf{d}^{H}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right).$$

■

7.3. *Show that the DF can be written as*

$$\mathcal{D}\left[\mathbf{h}(f)\right] = \mathcal{D}_{\max}\left(f,\cos\theta_{\mathrm{d}}\right)\cos^{2}\left[\mathbf{\Gamma}_{0,\pi}^{-1/2}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right),\mathbf{\Gamma}_{0,\pi}^{1/2}(f)\mathbf{h}(f)\right].$$

Solution Recall that :

- $$\mathcal{D}_{\max}\left(f,\cos\theta_{\mathrm{d}}\right) = \mathbf{d}^{H}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right).$$
- $$\mathcal{D} = \frac{\left|\mathbf{h}^{H}(f)\mathbf{d}(f,\cos\theta_{d})\right|^{2}}{\mathbf{h}^{H}(f)\mathbf{\Gamma}_{0,\pi}(f)\mathbf{h}(f)}.$$
- $$\begin{aligned}&\cos\left[\mathbf{\Gamma}_{0,\pi}^{-1/2}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right),\mathbf{\Gamma}_{0,\pi}^{1/2}(f)\mathbf{h}(f)\right] \\ &= \frac{\mathbf{d}^{H}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{h}(f) + \mathbf{h}^{H}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)}{2\left|\left|\mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\right|\right|_{2}\left|\left|\mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f)\mathbf{h}\left(f\right)\right|\right|_{2}}.\end{aligned}$$
- The distortionless constraint,

$$\mathbf{h}^{H}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right) = 1.$$

We will show that the wanted form is equivalent to the second bullet. Under the distortionless constraint, we have,

$$\cos\left[\mathbf{\Gamma}_{0,\pi}^{-1/2}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right),\mathbf{\Gamma}_{0,\pi}^{1/2}(f)\mathbf{h}(f)\right]=\frac{1}{\left|\left|\mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\right|\right|_2\left|\left|\mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f)\mathbf{h}\left(f\right)\right|\right|_2},$$

$$\mathcal{D}\left[\mathbf{h}(f)\right]=\frac{1}{\mathbf{h}^H(f)\mathbf{\Gamma}_{0,\pi}(f)\mathbf{h}(f)},$$

$$\begin{aligned}
&\mathcal{D}_{\max}\left(f,\cos\theta_{\mathrm{d}}\right)\cos^2\left[\mathbf{\Gamma}_{0,\pi}^{-1/2}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right),\mathbf{\Gamma}_{0,\pi}^{1/2}(f)\mathbf{h}(f)\right]\\
&=\frac{\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)}{\left(\left|\left|\mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\right|\right|_2\left|\left|\mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f)\mathbf{h}\left(f\right)\right|\right|_2\right)^2}\\
&=\frac{\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)}{\left(\mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\right)^H\mathbf{\Gamma}_{0,\pi}^{-\frac{1}{2}}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\left(\mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f)\mathbf{h}\left(f\right)\right)^H\mathbf{\Gamma}_{0,\pi}^{\frac{1}{2}}(f)\mathbf{h}\left(f\right)}\\
&=\frac{\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)}{\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{\Gamma}_{0,\pi}^{-1}(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{h}^H\left(f\right)\mathbf{\Gamma}_{0,\pi}(f)\mathbf{h}\left(f\right)}\\
&=\frac{1}{\mathbf{h}^H(f)\mathbf{\Gamma}_{0,\pi}(f)\mathbf{h}(f)}\\
&=\mathcal{D}\left[\mathbf{h}(f)\right].
\end{aligned}$$

■

7.4. *Show that the condition to prevent spatial aliasing is*

$$\frac{\delta}{\lambda}<\frac{1}{2}.$$

Solution We have,

$$\frac{\delta}{\lambda}=\frac{1}{cos(\theta_1)-cos(\theta_2)},$$

where $\theta_1\neq\theta_2$ and $\lambda,\delta>0$, by definition. Namely, to prevent spatial aliasing, one must demand :

$$0<\frac{\delta}{\lambda}<\frac{1}{cos(\theta_1)-cos(\theta_2)}=\frac{1}{|cos(\theta_1)-cos(\theta_2)|},$$

where the last inequality holds, again, by definition. Over the unique range of $0\leq\theta_1,\theta_2\leq\pi$, and specifically where $cos(\theta_1)-cos(\theta_2)>0$, we have,

$$\begin{aligned}
&|cos(\theta_1)-cos(\theta_2)|\leq 2\\
&\frac{1}{|cos(\theta_1)-cos(\theta_2)|}\geq\frac{1}{2}.
\end{aligned}$$

Hence,

$$\frac{\delta}{\lambda} < \frac{1}{2}.$$

■

7.5. *Show that the DS beamformer :*

$$\mathbf{h}_{\mathrm{DS}}\left(f, \cos\theta_{\mathrm{d}}\right) = \frac{\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)}{M},$$

maximizes the WNG, i.e.,

$$\min_{\mathbf{h}(f)} \mathbf{h}^H(f)\mathbf{h}(f) \quad \text{subject to} \quad \mathbf{h}^H(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) = 1.$$

Solution Let us define the Lagrangian, with the Lagrange multiplier, λ :

$$\mathcal{L}\left(\mathbf{h}, \lambda\right) = \mathbf{h}^H(f)\mathbf{h}(f) + \lambda\left(\mathbf{h}^H(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) - 1\right).$$

Now,

$$\begin{aligned}
\frac{\partial \mathcal{L}\left(\mathbf{h}, \lambda\right)}{\partial \mathbf{h}^H}\left(\mathbf{h}_{\mathrm{DS}}, \lambda\right) &= \mathbf{0} \\
2\mathbf{h}_{\mathrm{DS}}\left(f, \cos\theta_{\mathrm{d}}\right) + \lambda\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) &= \mathbf{0}, \text{ and} \\
\frac{\partial \mathcal{L}\left(\mathbf{h}, \lambda\right)}{\partial \lambda}\left(\mathbf{h}_{\mathrm{DS}}, \lambda\right) &= 0 \\
\mathbf{h}_{\mathrm{DS}}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) &= 1.
\end{aligned}$$

Now, we have,

$$\mathbf{h}_{\mathrm{DS}}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) = \frac{\mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)}{M}\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right).$$

Now,

$$\frac{\mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)}{M} = \frac{1}{M}\sum_{m=0}^{M} e^{-jm2\pi f\tau_0\cos\theta}e^{jm2\pi f\tau_0\cos\theta} = \frac{M}{M} = 1.$$

Thus,

$$\mathbf{h}_{\mathrm{DS}}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) = 1.$$

Now,

$$\begin{aligned}
2\mathbf{h}_{\mathrm{DS}}\left(f, \cos\theta_{\mathrm{d}}\right) + \lambda\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) &= \mathbf{0} \\
2\mathbf{h}_{\mathrm{DS}}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{h}_{\mathrm{DS}}\left(f, \cos\theta_{\mathrm{d}}\right) + \lambda\mathbf{h}_{\mathrm{DS}}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) &= 0.
\end{aligned}$$

Hence,

$$\lambda = -2\mathbf{h}_{\mathrm{DS}}^H (f, \cos\theta_{\mathrm{d}})\, \mathbf{h}_{\mathrm{DS}} (f, \cos\theta_{\mathrm{d}}) .$$

Hence,

$$\mathbf{h}_{\mathrm{DS}}^H (f, \cos\theta_{\mathrm{d}})\, \mathbf{h}_{\mathrm{DS}} (f, \cos\theta_{\mathrm{d}})\, \mathbf{d} (f, \cos\theta_{\mathrm{d}}) = \mathbf{h}_{\mathrm{DS}} (f, \cos\theta_{\mathrm{d}}) .$$

Again,

$$\frac{\mathbf{d}^H (f, \cos\theta_{\mathrm{d}})}{M} \frac{\mathbf{d} (f, \cos\theta_{\mathrm{d}})}{M} \mathbf{d} (f, \cos\theta_{\mathrm{d}}) = \frac{\mathbf{d} (f, \cos\theta_{\mathrm{d}})}{M},$$

Thus, it was shown that $\mathbf{h}_{\mathrm{DS}} (f, \cos\theta_{\mathrm{d}}) = \frac{\mathbf{d}(f, \cos\theta_{\mathrm{d}})}{M}$ indeed maximizes the WNG.

■

7.6. *Show that with the DS beamformer, the DF is given by*

$$\mathcal{D} [\mathbf{h}_{\mathrm{DS}} (f, \cos\theta_{\mathrm{d}})] = \frac{M^2}{\mathbf{d}^H (f, \cos\theta_{\mathrm{d}})\, \mathbf{\Gamma}_{0,\pi}(f) \mathbf{d} (f, \cos\theta_{\mathrm{d}})} .$$

Solution We have,

$$\mathcal{D} [\mathbf{h} (f)] = \frac{|\mathbf{h}^H (f)\, \mathbf{d} (f, \cos\theta_d)\, |^2}{\mathbf{h}^H (f)\, \mathbf{\Gamma}_{0,\pi} (f)\, \mathbf{h} (f)},$$

Hence,

$$\begin{aligned}
\mathcal{D} [\mathbf{h}_{\mathrm{DS}} (f, \cos\theta_{\mathrm{d}})] &= \frac{|\mathbf{h}_{\mathrm{DS}}^H (f, \cos\theta_{\mathrm{d}})\, \mathbf{d} (f, \cos\theta_d)\, |^2}{\mathbf{h}_{\mathrm{DS}}^H (f, \cos\theta_{\mathrm{d}})\, \mathbf{\Gamma}_{0,\pi} (f)\, \mathbf{h}_{\mathrm{DS}} (f, \cos\theta_{\mathrm{d}})} \\
&= \frac{|\frac{\mathbf{d}^H (f, \cos\theta_{\mathrm{d}})}{M} \mathbf{d} (f, \cos\theta_d)\, |^2}{\frac{\mathbf{d}^H (f, \cos\theta_{\mathrm{d}})}{M} \mathbf{\Gamma}_{0,\pi} (f)\, \frac{\mathbf{d}(f, \cos\theta_{\mathrm{d}})}{M}} \\
&= \frac{1}{\frac{\mathbf{d}^H (f, \cos\theta_{\mathrm{d}})}{M} \mathbf{\Gamma}_{0,\pi} (f)\, \frac{\mathbf{d}(f, \cos\theta_{\mathrm{d}})}{M}} \\
&= \frac{M^2}{\mathbf{d}^H (f, \cos\theta_{\mathrm{d}})\, \mathbf{\Gamma}_{0,\pi}(f) \mathbf{d} (f, \cos\theta_{\mathrm{d}})} .
\end{aligned}$$

■

7.7. *Show that the DS beamformer never amplifies the diffuse noise, i.e.,*

$$\mathcal{D} [\mathbf{h}_{\mathrm{DS}} (f, \cos\theta_{\mathrm{d}})] \geq 1.$$

8
Adaptive Beamforming

Problems

8.1. *Show that the narrowband MSE can be expressed as*

$$J\left[\mathbf{h}(f)\right] = \phi_X(f) + \mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{y}}(f)\mathbf{h}(f) - \phi_X(f)\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right) - \phi_X(f)\mathbf{d}^H\left(f,\cos\theta_{\mathrm{d}}\right)\mathbf{h}(f).$$

Solution The error signal between the estimated and desired signals is given by,

$$\begin{aligned}\mathcal{E}(f) &= Z(f) - X(f)\\ &= \left[\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right) - 1\right]X(f) + \mathbf{h}^H(f)\mathbf{v}(f)\\ &= \mathcal{E}_{\mathrm{d}}(f) + \mathcal{E}_{\mathrm{n}}(f),\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_{\mathrm{d}}(f) &= \left[\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\mathrm{d}}\right) - 1\right]X(f), \text{ and}\\ \mathcal{E}_{\mathrm{n}}(f) &= \mathbf{h}^H(f)\mathbf{v}(f),\end{aligned}$$

are the errors due to the desired signal distortion and residual noise, respectively. Since $\mathcal{E}_{\mathrm{d}}(f)$ and $\mathcal{E}_{\mathrm{n}}(f)$ are assumed to be incoherent, the narrowband MSE of the error signal becomes the sum of the narrowband MSEs of the desired signal distortion and the residual noise :

$$J\left[\mathbf{h}(f)\right] = \mathbb{E}\left[\left|\mathcal{E}_{\mathrm{d}}(f)\right|^2\right] + \mathbb{E}\left[\left|\mathcal{E}_{\mathrm{n}}(f)\right|^2\right].$$

As a result,

$$\begin{aligned}
&\mathbb{E}\left[|\mathcal{E}_{\rm d}(f)|^2\right] \\
&= \mathbb{E}\left[\mathcal{E}_{\rm d}(f)\mathcal{E}_{\rm d}^*(f)\right] \\
&= \mathbb{E}\left[\left[\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\rm d}\right)-1\right]X(f)X^*(f)\left[\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\rm d}\right)-1\right]^H\right] \\
&= \left[\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\rm d}\right)-1\right]\phi_X(f)\left[\mathbf{d}^H\left(f,\cos\theta_{\rm d}\right)\mathbf{h}(f)-1\right] \\
&= \phi_X(f)\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\rm d}\right)\mathbf{d}^H\left(f,\cos\theta_{\rm d}\right)\mathbf{h}(f)-\phi_X(f)\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\rm d}\right) \\
&\quad -\phi_X(f)\mathbf{d}^H\left(f,\cos\theta_{\rm d}\right)\mathbf{h}(f)+\phi_X(f) \\
&= \mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{x}}(f)\mathbf{h}(f)-\phi_X(f)\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\rm d}\right) \\
&\quad -\phi_X(f)\mathbf{d}^H\left(f,\cos\theta_{\rm d}\right)\mathbf{h}(f)+\phi_X(f),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[|\mathcal{E}_{\rm n}(f)|^2\right] &= \mathbb{E}\left[\mathcal{E}_{\rm n}(f)\mathcal{E}_{\rm n}^*(f)\right] \\
&= \mathbb{E}\left[\mathbf{h}^H(f)\mathbf{v}(f)\mathbf{v}^H(f)\mathbf{h}(f)\right] \\
&= \mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{v}}(f)\mathbf{h}(f).
\end{aligned}$$

We finally obtain :

$$\begin{aligned}
J\left[\mathbf{h}(f)\right] = &\, \mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{y}}(f)\mathbf{h}(f)-\phi_X(f)\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\rm d}\right) \\
&-\phi_X(f)\mathbf{d}^H\left(f,\cos\theta_{\rm d}\right)\mathbf{h}(f)+\phi_X(f).
\end{aligned}$$

■

8.2. *Show that the MSEs are related to the different performance measures by*

$$\begin{aligned}
\frac{J_{\rm d}\left[\mathbf{h}(f)\right]}{J_{\rm n}\left[\mathbf{h}(f)\right]} &= \mathrm{iSNR}(f)\times\xi_{\rm n}\left[\mathbf{h}(f)\right]\times\upsilon_{\rm d}\left[\mathbf{h}(f)\right] \\
&= \mathrm{oSNR}\left[\mathbf{h}(f)\right]\times\xi_{\rm d}\left[\mathbf{h}(f)\right]\times\upsilon_{\rm d}\left[\mathbf{h}(f)\right].
\end{aligned}$$

Solution Recall that

$$\mathrm{iSNR}(f) = \frac{\phi_X(f)}{\phi_{V_1}(f)}$$

Recall that the desired-signal and residual noise MSEs are :

$$\begin{aligned}
J_{\rm d}[\mathbf{h}(f)] &= \phi_X(f)\left|\mathbf{h}^H(f)\mathbf{d}\left(f,\cos\theta_{\rm d}\right)-1\right|^2 \\
&= \phi_X(f)\upsilon_{\rm d}[\mathbf{h}(f)], \\
J_{\rm n}[\mathbf{h}(f)] &= \mathbf{h}^H(f)\mathbf{\Phi}_{\rm v}(f)\mathbf{h}(f) \\
&= \frac{\phi_{V_1}(f)}{\xi_{\rm n}[\mathbf{h}(f)]}.
\end{aligned}$$

Recall that

$$\frac{\mathrm{oSNR}[\mathbf{h}(f)]}{\mathrm{iSNR}(f)} = \frac{\xi_{\mathrm{n}}[\mathbf{h}(f)]}{\xi_{\mathrm{d}}[\mathbf{h}(f)]}.$$

Hence, we have,

$$\begin{aligned}\frac{J_{\mathrm{d}}\left[\mathbf{h}(f)\right]}{J_{\mathrm{n}}\left[\mathbf{h}(f)\right]} &= \frac{\phi_X(f)\upsilon_{\mathrm{d}}[\mathbf{h}(f)]}{\frac{\phi_{V_1}(f)}{\xi_{\mathrm{n}}[\mathbf{h}(f)]}} \\ &= \mathrm{iSNR}(f) \times \xi_{\mathrm{n}}\left[\mathbf{h}(f)\right] \times \upsilon_{\mathrm{d}}\left[\mathbf{h}(f)\right] \\ &= \mathrm{oSNR}\left[\mathbf{h}(f)\right] \times \xi_{\mathrm{d}}\left[\mathbf{h}(f)\right] \times \upsilon_{\mathrm{d}}\left[\mathbf{h}(f)\right].\end{aligned}$$

■

8.3. *Show that by minimizing the narrowband MSE, $J\left[\mathbf{h}(f)\right]$, we obtain the Wiener beamformer:*

$$\mathbf{h}_{\mathrm{W}}\left(f, \cos\theta_{\mathrm{d}}\right) = \phi_X(f)\mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right).$$

Solution The narrowband MSE is given by,

$$\begin{aligned}J[\mathbf{h}(f)] &= E\left[|\mathcal{E}(f)|^2\right] \\ &= E\left[\left|\mathcal{E}_{\mathrm{d}}(f)\right|^2\right] + E\left[\left|\mathcal{E}_{\mathrm{n}}(f)\right|^2\right] \\ &= J_{\mathrm{d}}[\mathbf{h}(f)] + J_{\mathrm{n}}[\mathbf{h}(f)] \\ &= \phi_X(f) + \mathbf{h}^H(f)\mathbf{\Phi}_{\mathbf{y}}(f)\mathbf{h}(f) - \phi_X(f)\mathbf{h}^H(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) \\ &\quad - \phi_X(f)\mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{h}(f).\end{aligned}$$

Hence, we have,

$$\begin{aligned}\nabla_{\mathbf{h}^*(f)} J\left[\mathbf{h}(f)\right] &= \mathbf{0} \\ 2\mathbf{\Phi}_{\mathbf{y}}(f)\mathbf{h}(f) - \phi_{X_1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) - \phi_{X_1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) &= \mathbf{0}.\end{aligned}$$

Thus,

$$\mathbf{h}_{\mathrm{W}}\left(f, \cos\theta_{\mathrm{d}}\right) = \phi_{X_1}(f)\mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right).$$

■

8.4. *Show that the Wiener beamformer can be written as*

$$\mathbf{h}_{\mathrm{W}}\left(f, \cos\theta_{\mathrm{d}}\right) = \frac{\mathrm{iSNR}(f)}{1 + \mathrm{iSNR}(f)}\mathbf{\Gamma}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right).$$

Solution Since $\mathbf{x}(f)$ and $\mathbf{v}(f)$ are incoherent and stationary, for $m = 1, ..., M$, we have

$$\begin{aligned}\phi_Y(f) &= \mathbb{E}\left[|Y(f)|^2\right] = \mathbb{E}\left[|X(f)|^2\right] + \mathbb{E}\left[|V(f)|^2\right] \\ &= \phi_X(f) + \phi_V(f).\end{aligned}$$

Additionally, since

$$\mathbf{\Gamma}_{\mathbf{y}}(f) = \frac{\mathbf{\Phi}_{\mathbf{y}}(f)}{\phi_Y(f)},$$

we simplify the Wiener filter as

$$\begin{aligned}\mathbf{h}_{\mathrm{W}}\left(f, \cos\theta_{\mathrm{d}}\right) &= \phi_X(f)\mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) \\ &= \frac{\phi_X(f)}{\phi_X(f) + \phi_V(f)}\left[\phi_X(f) + \phi_V(f)\right]\mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) \\ &= \frac{\frac{\phi_X(f)}{\phi_V(f)}}{\frac{\phi_X(f)}{\phi_V(f)} + 1}\mathbf{\Gamma}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) \\ &= \frac{\mathrm{iSNR}(f)}{1 + \mathrm{iSNR}(f)}\mathbf{\Gamma}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right).\end{aligned}$$

■

8.5. *Show that the Wiener beamformer can be expressed as a function of the statistics of the observation and noise signals by*

$$\mathbf{h}_{\mathrm{W}}\left(f, \cos\theta_{\mathrm{d}}\right) = \left[\mathbf{I}_M - \mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{\Phi}_{\mathbf{v}}(f)\right]\mathbf{i}_{\mathrm{i}}.$$

Solution Using the following equality,

$$\mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{i}_{\mathrm{i}} = 1,$$

we can rewrite the Wiener filter as,

$$\begin{aligned}\mathbf{h}_{\mathrm{W}}\left(f, \cos\theta_{\mathrm{d}}\right) &= \phi_X(f)\mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right) \\ &= \phi_X(f)\mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{i}_{\mathrm{i}} \\ &= \mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{\Phi}_{\mathbf{x}}(f)\mathbf{i}_{\mathrm{i}} \\ &= \mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\left[\mathbf{\Phi}_{\mathbf{y}}(f) - \mathbf{\Phi}_{\mathbf{v}}(f)\right]\mathbf{i}_{\mathrm{i}} \\ &= \left[\mathbf{I}_M - \mathbf{\Phi}_{\mathbf{y}}^{-1}(f)\mathbf{\Phi}_{\mathbf{v}}(f)\right]\mathbf{i}_{\mathrm{i}}.\end{aligned}$$

■

8.6. *Using the Woodbury's identity, show that the inverse of $\mathbf{\Phi}_{\mathbf{y}}(f)$ is given by*

$$\mathbf{\Phi}_{\mathbf{y}}^{-1}(f) = \mathbf{\Phi}_{\mathbf{v}}^{-1}(f) - \frac{\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)}{\phi_X^{-1}(f) + \mathbf{d}^H\left(f, \cos\theta_{\mathrm{d}}\right)\mathbf{\Phi}_{\mathbf{v}}^{-1}(f)\mathbf{d}\left(f, \cos\theta_{\mathrm{d}}\right)}.$$

9
Differential Beamforming

Problems

9.1. *Using the definition of the frequency-independent DF of a theoretical Nth-order DSA (**??**), show that*

$$\mathcal{D}(\mathbf{a}_N) = \frac{\mathbf{a}_N^T \mathbf{1}\mathbf{1}^T \mathbf{a}_N}{\mathbf{a}_N^T \mathbf{H}_N \mathbf{a}_N},$$

where $\mathbf{1}$ is a vector of ones, and $\mathbf{H}_N$ is a Hankel matrix.

Solution Recall that

$$\mathcal{D}(\mathbf{a}_N) = \frac{\mathcal{B}^2(\mathbf{a}_N, 1)}{\frac{1}{2}\int_0^\pi \mathcal{B}^2(\mathbf{a}_N, \cos\theta)\sin\theta d\theta}.$$

Using the definition of $\mathcal{B}(\mathbf{a}_N, \cos\theta)$, we have,

$$\begin{aligned}\mathcal{D}(\mathbf{a}_N) &= \frac{\left[\sum_{n=0}^N a_{N,n}\right]^2}{\frac{1}{2}\int_0^\pi \left[\sum_{n=0}^N a_{N,n}\cos^n\theta\right]^2 \sin\theta d\theta}. \\ &= \frac{\left[\sum_{n=0}^N a_{N,n}\right]^2}{\frac{1}{2}\int_0^\pi \left[\sum_{n=0}^N a_{N,n}^2 \cos^{2n}\theta + \sum_{i=0}^N \sum_{j=0, j\neq i}^N a_{N,i}a_{N,j}\cos^{i+j}\theta\right]\sin\theta d\theta},\end{aligned}$$

which can be reformulated due to the linearity of the integral and summation operations :

$$\begin{aligned}&\mathcal{D}(\mathbf{a}_N) \\ &= \frac{\left[\sum_{n=0}^N a_{N,n}\right]^2}{\frac{1}{2}\left[\sum_{n=0}^N a_{N,n}^2 \int_0^\pi \cos^{2n}\theta\sin\theta d\theta\right] + \frac{1}{2}\left[\sum_{i=0}^N \sum_{j=0, j\neq i}^N a_{N,i}a_{N,j}\int_0^\pi \cos^{i+j}\theta\sin\theta d\theta\right]} \\ &= \frac{\left[\sum_{n=0}^N a_{N,n}\right]^2}{\sum_{n=0}^N \frac{a_{N,n}^2}{n+1} + \sum_{i=0}^N \sum_{j=0, j\neq i, i+j \text{ even}}^N \frac{a_{N,i}a_{N,j}}{i+j+1}}.\end{aligned}$$

Adopting the definitions of the Hankel matrix, and $\mathbf{a}_N$, we have,

$$\mathcal{D}\left(\mathbf{a}_N\right) = \frac{\mathbf{a}_N^T \mathbf{1}\mathbf{1}^T \mathbf{a}_N}{\mathbf{a}_N^T \mathbf{H}_N \mathbf{a}_N}.$$

■

9.2. *Show that the coefficients of the Nth-order hypercardioid are given by*

$$\mathbf{a}_{N,\max} = \frac{\mathbf{H}_N^{-1}\mathbf{1}}{\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}}.$$

Solution According to its definition, $\mathbf{a}_{N,\max}$ is the right eigenvector corresponding to the maximal eigenvalue of the matrix $\mathbf{H}_N^{-1}\mathbf{1}\mathbf{1}^T$. According to eq. (10.62) in the book "Superdirectional Microphone Arrays", this eigenvalue is equal to $\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}$. Namely, what we would like to prove is that the following holds :

$$\mathbf{H}_N^{-1}\mathbf{1}\mathbf{1}^T \mathbf{a}_{N,\max} = \mathbf{a}_{N,\max}\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}.$$

By substituting the desired expression in the above, one yields :

$$\begin{aligned} \mathbf{H}_N^{-1}\mathbf{1}\mathbf{1}^T \frac{\mathbf{H}_N^{-1}\mathbf{1}}{\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}} &= \frac{\mathbf{H}_N^{-1}\mathbf{1}}{\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}}\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1} \\ \frac{\mathbf{H}_N^{-1}\mathbf{1}\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}}{\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}} &= \frac{\mathbf{H}_N^{-1}\mathbf{1}\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}}{\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}}, \end{aligned}$$

which validates the assumption that $\mathbf{a}_{N,\max} = \frac{\mathbf{H}_N^{-1}\mathbf{1}}{\mathbf{1}^T \mathbf{H}_N^{-1}\mathbf{1}}$.

■

9.3. *Using the definition of the frequency-independent FBR of a theoretical Nth-order DSA (**??**), show that*

$$\mathcal{F}\left(\mathbf{a}_N\right) = \frac{\mathbf{a}_N^T \mathbf{H}_N'' \mathbf{a}_N}{\mathbf{a}_N^T \mathbf{H}_N' \mathbf{a}_N},$$

where $\mathbf{H}_N'$ and $\mathbf{H}_N''$ are Hankel matrices.

Solution We have,

$$
\begin{aligned}
&\mathcal{F}(\mathbf{a}_N) \\
&= \frac{\int_0^{\pi/2} \mathcal{B}^2(\mathbf{a}_N, \cos\theta)\sin\theta d\theta}{\int_{\pi/2}^{\pi} \mathcal{B}^2(\mathbf{a}_N, \cos\theta)\sin\theta d\theta} \\
&= \frac{\int_0^{\pi/2} \left[\sum_{n=0}^N a_{N,n}\cos^n\theta\right]^2 \sin\theta d\theta}{\int_{\pi/2}^{\pi} \left[\sum_{n=0}^N a_{N,n}\cos^n\theta\right]^2 \sin\theta d\theta} \\
&= \frac{\left[\sum_{n=0}^N a_{N,n}^2 \int_0^{\pi/2}\cos^{2n}\theta\sin\theta d\theta\right] + \left[\sum_{i=0}^N\sum_{j=0,j\neq i}^N a_{N,i}a_{N,j}\int_0^{\pi/2}\cos^{i+j}\theta\sin\theta d\theta\right]}{\left[\sum_{n=0}^N a_{N,n}^2 \int_{\pi/2}^{\pi}\cos^{2n}\theta\sin\theta d\theta\right] + \left[\sum_{i=0}^N\sum_{j=0,j\neq i}^N a_{N,i}a_{N,j}\int_{\pi/2}^{\pi}\cos^{i+j}\theta\sin\theta d\theta\right]} \\
&= \frac{\left[\sum_{n=0}^N a_{N,n}^2 \frac{1}{2n+1}\right] + \left[\sum_{i=0}^N\sum_{j=0,j\neq i}^N a_{N,i}a_{N,j}\frac{1}{i+j+1}\right]}{\left[\sum_{n=0}^N a_{N,n}^2 \frac{1}{2n+1}\right] + \left[\sum_{i=0}^N\sum_{j=0,j\neq i}^N a_{N,i}a_{N,j}\frac{(-1)^{i+j}}{i+j+1}\right]}.
\end{aligned}
$$

Hence, according to the definitions of the Hankel matrices, we have,

$$
\mathcal{F}(\mathbf{a}_N) = \frac{\mathbf{a}_N^T \mathbf{H}_N'' \mathbf{a}_N}{\mathbf{a}_N^T \mathbf{H}_N' \mathbf{a}_N}.
$$

■

9.4. *Show that the beampattern of the Nth-order supercardioid is*

$$
\mathcal{B}_{N,\mathrm{Sd}}(\cos\theta) = \frac{\mathbf{a}_{N,\max}'^T \mathbf{p}(\cos\theta)}{\mathbf{a}_{N,\max}'^T \mathbf{p}(1)},
$$

where $\mathbf{a}_{N,\max}'$ is the eigenvector corresponding to the maximum eigenvalue of $\mathbf{H}_N'^{-1}\mathbf{H}_N''$.

Solution First, we would like to show that $\mathbf{H}_N'$ is positive-definite and that $\mathbf{H}_N''$ is positive semi-definite. Assuming a non-zero vector, $\mathbf{x} \in \mathbb{R}^{N\times 1}$:

$$
\begin{aligned}
\mathbf{x}^H \mathbf{H}_N'' \mathbf{x} &= \sum_{i=0}^N \sum_{j=0}^N \overline{x_i} x_j \frac{1}{i+j+1} \\
&= \sum_{i=0}^N \sum_{j=0}^N \overline{x_i} x_j \int_{t=0}^1 t^{i+j} dt \\
&= \int_{t=0}^1 \overline{\sum_{i=0}^N t^i x_i} \sum_{j=0}^N t^j x_j dt \\
&= \int_{t=0}^1 |\sum_{i=0}^N t^i x_i|^2 dt \geq 0,
\end{aligned}
$$

which proves that $\mathbf{H}''_N$ is a positive semi-definite matrix.
Similarly,

$$\begin{aligned}\mathbf{x}^H\mathbf{H}'_N\mathbf{x} &= \sum_{i=0}^{N}\sum_{j=0}^{N}\overline{x_i}x_j\frac{(-1)^{i+j}}{i+j+1} \\ &= -\sum_{i=0}^{N}\sum_{j=0}^{N}\overline{x_i}x_j\int_{t=0}^{-1}t^{i+j}dt \\ &= \int_{t=-1}^{0}\overline{\sum_{i=0}^{N}t^i x_i}\sum_{j=0}^{N}t^j x_j dt \\ &= \int_{t=-1}^{0}|\sum_{i=0}^{N}t^i x_i|^2 dt > 0,\end{aligned}$$

which entails that $\mathbf{H}'_N$ is strictly positive matrix. It should be highlighted that the latter inequality is true, since for small enough $\epsilon > 0$, the following holds :

$$|\sum_{i=0}^{N}t^i x_i|^2\Big|_{t=-\epsilon} \approx |-x_k\epsilon^k|^2 > 0,$$

where k is the first non-zero element of $\mathbf{x}$.
Conclusively, the following expression is indeed a Rayleigh quotient.

$$\mathcal{F}(\mathbf{a}_N) = \frac{\mathbf{a}_N^T\mathbf{H}''_N\mathbf{a}_N}{\mathbf{a}_N^T\mathbf{H}'_N\mathbf{a}_N},$$

Thus, as stated, the vector that maximizes the above expression is equal to eigenvector corresponding to the maximal eigenvalue of $\mathbf{H}'^{-1}_N\mathbf{H}''_N$, which is $\mathbf{a}'_{N,\text{max}}$. Notice that the coefficients vector must satisfy eq. (9.8), i.e., $\mathbf{a}'^T_{N,\text{max}}$ must be normalized to $\frac{\mathbf{a}'^T_{N,\text{max}}}{\sum_{i=0}^{N}\mathbf{a}'^T_{N,\text{max}}(i)} = \frac{\mathbf{a}'^T_{N,\text{max}}}{\mathbf{a}'^T_{N,\text{max}}\mathbf{p}(1)}$, where the last equality employs the definitions given in eq. (9.7). Ultimately, this eigenvector can be replaced into eq. (9.7) to yield the desired outcome :

$$\mathcal{B}_{N,\text{Sd}}(\cos\theta) = \frac{\mathbf{a}'^T_{N,\text{max}}\mathbf{p}(\cos\theta)}{\mathbf{a}'^T_{N,\text{max}}\mathbf{p}(1)}.$$

■

9.5. *Show that the directivity pattern of the first-order hypercardioid can be expressed as,*

$$\mathcal{B}_{1,\text{Hd}}(\cos\theta) = \frac{1}{4} + \frac{3}{4}\cos\theta.$$

Solution Let us refer to the general definition of $\mathcal{B}_{N,H_d}(\cos\theta)$ according to eq. (9.34), with $N = 1$. Namely :

$$\mathcal{B}_{1,\mathrm{H}_d}(\cos\theta) = \mathcal{B}_{N,\mathrm{H}_d}(\cos\theta)\big|_{N=1} = \frac{\mathbf{1}^T\mathbf{H}_N^{-1}\mathbf{p}(\cos\theta)}{\mathbf{1}^T\mathbf{H}_N^{-1}\mathbf{1}}\Big|_{N=1}.$$

Next, we employ the definitions of the Hankel matrix, $\mathbf{H}_N$, given in eq. (9.32), and $\mathbf{p}(\cos\theta)$, given in (9.7), to obtain :

$$\mathcal{B}_{1,\mathrm{H}_d}(\cos\theta) = \frac{\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1\\ \cos\theta\end{bmatrix}}{\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}} = \frac{1}{4} + \frac{3\cos\theta}{4}.$$

■

9.6. *Show that the directivity pattern of the first-order supercardioid can be expressed as*

$$\mathcal{B}_{1,\mathrm{Sd}}(\cos\theta) = \frac{\sqrt{3}-1}{2} + \frac{3-\sqrt{3}}{2}\cos\theta.$$

Solution Let us refer to the general definition of $\mathcal{B}_{N,S_d}(\cos\theta)$ according to eq. (9.38), with $N = 1$. Namely :

$$\mathcal{B}_{1,\mathrm{Sd}}(\cos\theta) = \mathcal{B}_{N,\mathrm{Sd}}(\cos\theta)\big|_{N=1} = \frac{\mathbf{a}_{N,\max}'^T\mathbf{p}(\cos\theta)}{\mathbf{a}_{N,\max}'^T\mathbf{p}(1)}\Big|_{N=1}.$$

Next, we employ the definition of $\mathbf{p}(\cos\theta)$, given in (9.7), to yield :

$$\mathcal{B}_{1,\mathrm{Sd}}(\cos\theta) = \frac{\begin{bmatrix}\mathbf{a}_{N,\max}'^T(0) & \mathbf{a}_{N,\max}'^T(1)\end{bmatrix}\begin{bmatrix}1\\ \cos\theta\end{bmatrix}}{\begin{bmatrix}\mathbf{a}_{N,\max}'^T(0) & \mathbf{a}_{N,\max}'^T(1)\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}} = \frac{\mathbf{a}_{N,\max}'^T(0) + \mathbf{a}_{N,\max}'^T(1)\cos\theta}{\mathbf{a}_{N,\max}'^T(0) + \mathbf{a}_{N,\max}'^T(1)}.$$

Next, it remains to find $\mathbf{a}_{N,\max}'^T$, which is now defined as the **normalized** eigenvector, i.e. $\mathbf{a}_{N,\max}'^T = \frac{\mathbf{a}_{N,\max}'^T}{\mathbf{a}_{N,\max}'^T\mathbf{p}(1)}$, that corresponds to the largest eigenvalue of $\mathbf{H}_N'^{-1}\mathbf{H}_N''$. In this case :

$$\mathbf{H}_N'^{-1}\mathbf{H}_N'' = \begin{bmatrix}1 & -1/2\\ -1/2 & 1/3\end{bmatrix}^{-1}\begin{bmatrix}1 & 1/2\\ 1/2 & 1/3\end{bmatrix} = \begin{bmatrix}7 & 4\\ 12 & 7\end{bmatrix},$$

from which, a simple eigenvalue decomposition process yields the maximum eigenvalue $\lambda_{\max} \approx 13.92$ and its corresponding **normalized** eigenvector $\psi_{\max}^T = \mathbf{a}_{N,\max}'^T = \frac{1}{2}\begin{bmatrix}\sqrt{3}-1, & 3-\sqrt{3}\end{bmatrix}$. Ultimately, we have,

$$\mathcal{B}_{1,\mathrm{Sd}}(\cos\theta) = \frac{\sqrt{3}-1+\left(3-\sqrt{3}\right)\cos\theta}{2} = \frac{\sqrt{3}-1}{2} + \frac{3-\sqrt{3}}{2}\cos\theta.$$

10
Beampattern Design

Problems

10.1. *Show that the minimization of the LSE criterion yields*

$$\mathbf{c}_N = \mathbf{M}_{\mathrm{C}}^{-1}\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right).$$

Solution Let us recall the definition of the LSE criterion, given in eq. (10.12) :

$$\mathrm{LSE}(\mathbf{c}_N) = 1 - \mathbf{v}_{\mathrm{C}}^H\left(\jmath\overline{f}_m\right)\mathbf{c}_N - \mathbf{c}_N^H\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right) + \mathbf{c}_N^H\mathbf{M}_{\mathrm{C}}\mathbf{c}_N,$$

Next, let us focus on the expressions $\mathbf{v}_{\mathrm{C}}^H\left(\jmath\overline{f}_m\right)\mathbf{c}_N$ and $\mathbf{c}_N^H\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)$, using the definition in eqs. (10.2, 10.6, 10.12, 10.14), from which we can derive that :

$$\begin{aligned}
\mathbf{v}_{\mathrm{C}}^H\left(\jmath\overline{f}_m\right)\mathbf{c}_N &= \frac{1}{\pi}\sum_{i=0}^{N}\mathbf{c}_i\int_{\theta=0}^{\pi} e^{-\jmath\overline{f}_m\cos\theta}\cos\left(i\theta\right)d\theta \\
&= \frac{1}{\pi}\sum_{i=0}^{N}\mathbf{c}_i\int_{\theta=0}^{\pi}\left[\sum_{n=0}^{N}\mathbf{c}_n^H\cos\left(n\theta\right)\right]\cos\left(i\theta\right)d\theta, \\
\mathbf{c}_N^H\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right) &= \frac{1}{\pi}\sum_{i=0}^{N}\mathbf{c}_i^H\int_{\theta=0}^{\pi} e^{\jmath\overline{f}_m\cos\theta}\cos\left(i\theta\right)d\theta \\
&= \frac{1}{\pi}\sum_{i=0}^{N}\mathbf{c}_i^H\int_{\theta=0}^{\pi}\left[\sum_{n=0}^{N}\mathbf{c}_n\cos\left(n\theta\right)\right]\cos\left(i\theta\right)d\theta, \\
\frac{\partial\mathbf{v}_{\mathrm{C}}^H\left(\jmath\overline{f}_m\right)\mathbf{c}_N}{\partial\mathbf{c}_N} &= \frac{\partial\mathbf{c}_N^H\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)}{\partial\mathbf{c}_N}.
\end{aligned}$$

The optimal value of $\mathbf{c}_N$ which minimizes the LSE criterion is, therefore :

$$\begin{aligned}
\frac{\partial\mathrm{LSE}(\mathbf{c}_N)}{\partial\mathbf{c}_N} &= \mathbf{0} \\
\frac{\partial}{\partial\mathbf{c}_N}\left(1 - \mathbf{v}_{\mathrm{C}}^H\left(\jmath\overline{f}_m\right)\mathbf{c}_N - \mathbf{c}_N^H\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right) + \mathbf{c}_N^H\mathbf{M}_{\mathrm{C}}\mathbf{c}_N\right) &= \mathbf{0} \\
-2\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right) + 2\mathbf{M}_{\mathrm{C}}\mathbf{c}_N &= \mathbf{0} \\
\mathbf{c}_N &= \mathbf{M}_{\mathrm{C}}^{-1}\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right).
\end{aligned}$$

■

10.2. *Show that the elements of the vector* $\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)$ *are*

$$\left[\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)\right]_{n+1} = \jmath^n J_n\left(\overline{f}_m\right),$$

where $J_n(z)$ *is the Bessel function of the first kind.*

Solution Initially, let us recall the definition of the vector $\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)$, as introduces in eq. (10.12). By employing the definition of $\mathbf{p}_{\mathrm{C}}(\cos\theta)$ from eq. (10.6), the $(n+1)^{\text{th}}$ element of $\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)$ can be formulated as :

$$\left[\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)\right]_{n+1} = \frac{1}{\pi}\int_{\theta=0}^{\pi} e^{\jmath\overline{f}_m\cos\theta}\cos(n\theta)\,d\theta,$$

which is the exact representation in the first line of eq. (10.14). Next, let us use the definition of the modified Bessel function of the first kind, which according to eq. (10.16) is given by :

$$J_n(z) = \frac{\jmath^{-n}}{\pi}\int_{\theta=0}^{\pi} e^{\jmath z\cos\theta}\cos(n\theta)\,d\theta,\ \forall\, z\in\mathcal{C}.$$

Thus,

$$\left[\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)\right]_{n+1} = \frac{1}{\jmath^{-n}}\frac{\jmath^{-n}}{\pi}\int_{\theta=0}^{\pi} e^{\jmath z\cos\theta}\cos(n\theta)\,d\theta\Big|_{z=\overline{f}_m} = \jmath^n J_n(\overline{f}_m).$$

■

10.3. *Show that the elements of the matrix* $\mathbf{M}_{\mathrm{C}}$ *are*

$$[\mathbf{M}_{\mathrm{C}}]_{i+1,j+1} = \frac{1}{\pi}\int_0^{\pi}\cos(i\theta)\cos(j\theta)\,d\theta.$$

Solution Initially, let us recall the definition of $\mathbf{M}_{\mathrm{C}}$, given in eq. (10.12) :

$$\mathbf{M}_{\mathrm{C}} = \frac{1}{\pi}\int_{\theta=0}^{\pi}\mathbf{p}_{\mathrm{C}}(\cos\theta)\,\mathbf{p}_{\mathrm{C}}^T(\cos\theta)\,d\theta.$$

Let us focus on matrix generated by the following vector multiplication, defined as $\mathbf{P}$:

$$\mathbf{P} = \mathbf{p}_{\mathrm{C}}(\cos\theta)\,\mathbf{p}_{\mathrm{C}}^T(\cos\theta) \ \rightarrow [\mathbf{P}]_{i+1,j+1} = \cos(i\theta)\cos(j\theta).$$

Thus, we can formulate the $(i+1, j+1)^{\text{th}}$ element of $\mathbf{M}_{\mathrm{C}}$ as follows :

$$[\mathbf{M}_{\mathrm{C}}]_{i+1,j+1} = \frac{1}{\pi}\int_{\theta=0}^{\pi}[\mathbf{P}]_{i+1,j+1}\,d\theta = \frac{1}{\pi}\int_{\theta=0}^{\pi}\cos(i\theta)\cos(j\theta)\,d\theta.$$

■

10.4. *Prove the Jacobi-Anger expansion, i.e.,*

$$e^{\jmath \overline{f}_m \cos\theta} = \sum_{n=0}^{\infty} \jmath_n J_n\left(\overline{f}_m\right)\cos\left(n\theta\right),$$

where

$$\jmath_n = \begin{cases} 1, & n = 0 \\ 2\jmath^n, & n = 1, 2, \ldots, N \end{cases}.$$

Solution First, we recall the definition of $e^{\jmath \overline{f}_m \cos\theta}$, given in eq. (10.11) :

$$e^{\jmath \overline{f}_m \cos\theta} = \lim_{N\to\infty}\sum_{n=0}^{N} c_n \cos\left(n\theta\right).$$

In order to develop this expression, we should further expend the definition of c_n, $\forall\, 0 \leq n \leq N$. According to eq. (10.13), $\mathbf{c}_N = \mathbf{M}_{\mathrm{C}}^{-1}\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)$, so by replacing the definitions of $\mathbf{M}_{\mathrm{C}}$ and $\mathbf{v}_{\mathrm{C}}\left(\jmath\overline{f}_m\right)$ from eqs. (10.18) and (10.14), respectively, we can yield the vector $\mathbf{c}_N$:

$$\mathbf{c}_N = \mathrm{diag}\left(1, 2, \ldots, 2\right)\begin{bmatrix} J_0\left(\jmath\overline{f}_m\right) \\ \jmath J_1\left(\jmath\overline{f}_m\right) \\ \jmath^2 J_2\left(\jmath\overline{f}_m\right) \\ \cdots \\ \jmath^N J_N\left(\jmath\overline{f}_m\right) \end{bmatrix} = \begin{bmatrix} J_0\left(\jmath\overline{f}_m\right) \\ 2\jmath J_1\left(\jmath\overline{f}_m\right) \\ 2\jmath^2 J_2\left(\jmath\overline{f}_m\right) \\ \cdots \\ 2\jmath^N J_N\left(\jmath\overline{f}_m\right) \end{bmatrix},$$

where $\mathbf{c}_N = \left[c_0, c_1, \ldots, c_N\right]$. Now,

$$e^{\jmath \overline{f}_m \cos\theta} = J_0\left(\overline{f}_m\right) + \lim_{N\to\infty}\sum_{n=1}^{N} 2\jmath^n J_n\left(\overline{f}_m\right)\cos\left(n\theta\right).$$

By employing the definition of $\jmath_n$ introduced in eq. (10.19), we have,

$$\begin{aligned} e^{\jmath \overline{f}_m \cos\theta} &= \lim_{N\to\infty}\sum_{n=0}^{N} \jmath_n J_n\left(\overline{f}_m\right)\cos\left(n\theta\right) \\ &= \sum_{n=0}^{\infty} \jmath_n J_n\left(\overline{f}_m\right)\cos\left(n\theta\right). \end{aligned}$$

■

10.5. *Show that the beampattern can be approximated by*

$$\mathcal{B}_N\left[\mathbf{h}(f), \cos\theta\right] = \sum_{n=0}^{N}\cos\left(n\theta\right)\left[\sum_{m=1}^{M} \jmath_n J_n\left(\overline{f}_m\right) H_m(f)\right].$$

Solution Let us employ the definition of $\mathcal{B}_N\left[\mathbf{h}(f), \cos\theta\right]$ from eq. (10.1), and replace in it the expression for $e^{\jmath \overline{f}_m \cos\theta}$:

$$\begin{aligned}\mathcal{B}_N\left[\mathbf{h}(f), \cos\theta\right] &= \sum_{m=1}^{M} H_m(f) e^{\jmath \overline{f}_m \cos\theta} \\ &= \sum_{m=1}^{M} H_m(f) \sum_{n=0}^{\infty} \jmath_n J_n\left(\overline{f}_m\right) \cos\left(n\theta\right) \\ &= \sum_{n=0}^{\infty} \cos\left(n\theta\right) \left[\sum_{m=1}^{M} \jmath_n J_n\left(\overline{f}_m\right) H_m(f)\right],\end{aligned}$$

where the last equality is legitimate due to the additivity property. Now, by limiting n to a finite length, N, the former expression can be approximated as follows :

$$\mathcal{B}_N\left[\mathbf{h}(f), \cos\theta\right] = \sum_{n=0}^{N} \cos\left(n\theta\right) \left[\sum_{m=1}^{M} \jmath_n J_n\left(\overline{f}_m\right) H_m(f)\right].$$

■

10.6. *Show that with the nonrobust filter,* $\mathbf{h}_{\mathrm{NR}}(f)$*, the first-order beampattern is given by*

$$\mathcal{B}_1\left[\mathbf{h}(f), \cos\theta\right] = H_1(f) + J_0\left(\overline{f}_2\right) H_2(f) + 2\jmath J_1\left(\overline{f}_2\right) H_2(f)\cos\theta.$$

Solution Using equation (10.22) it can be stated that:

$$\mathcal{B}_1\left[\mathbf{h}(f), \cos\theta\right] = \sum_{i=0}^{1} \cos\left(i\theta\right) \overline{\mathbf{b}}_{\mathrm{i}}^T(f)\mathbf{h}(f) = \overline{\mathbf{b}}_0^T(f)\mathbf{h}(f) + \cos\theta \overline{\mathbf{b}}_1^T(f)\mathbf{h}(f).$$

Using equation (10.23) we find that :

$$\begin{aligned}&\mathcal{B}_1\left[\mathbf{h}(f), \cos\theta\right] \\ &= \jmath_0 \left[J_0\left(\overline{f}_1\right) \quad J_0\left(\overline{f}_2\right)\right] \begin{bmatrix} H_1(f) \\ H_2(f) \end{bmatrix} + \cos\theta \jmath_1 \left[J_1\left(\overline{f}_1\right) \quad J_1\left(\overline{f}_2\right)\right] \begin{bmatrix} H_1(f) \\ H_2(f) \end{bmatrix} \\ &= \jmath_0 J_0\left(\overline{f}_1\right) H_1(f) + \jmath_0 J_0\left(\overline{f}_2\right) H_2(f) + \cos\theta \jmath_1 J_1\left(\overline{f}_1\right) H_1(f) + \cos\theta \jmath_1 J_1\left(\overline{f}_2\right) H_2(f) \\ &= J_0\left(\overline{f}_1\right) H_1(f) + J_0\left(\overline{f}_2\right) H_2(f) + 2j\cos\theta J_1\left(\overline{f}_1\right) H_1(f) + 2j\cos\theta J_1\left(\overline{f}_2\right) H_2(f).\end{aligned}$$

Notice that from equation (10.2) it can be seen that $\overline{f}_1 = 0$. Therefore, from equation (10.16) it can be deduced that :

$$\begin{aligned}J_0\left(\overline{f}_1\right) &= \frac{1}{\pi}\int_0^{\pi} e^{j0\cos\theta} \cos\left(0\theta\right) d\theta = \frac{1}{\pi}\int_0^{\pi} 1 d\theta = 1, \\ J_1\left(\overline{f}_1\right) &= \frac{1}{\pi j}\int_0^{\pi} e^{j0\cos\theta} \cos\theta d\theta = \frac{1}{\pi j}\int_0^{\pi} \cos\theta d\theta = 0.\end{aligned}$$

11
Beamforming in the Time Domain

Problems

11.1. *Show that the MSE can be expressed as*

$$J\left(\underline{\mathbf{h}}\right) = \sigma_x^2 - 2\underline{\mathbf{h}}^T \underline{\mathbf{G}}\left(\cos\theta_{\mathrm{d}}\right)\mathbf{R}_{\mathbf{x}}\mathbf{i}_l + \underline{\mathbf{h}}^T \mathbf{R}_{\underline{\mathbf{y}}}\underline{\mathbf{h}}.$$

Solution The error signal between the estimated and desired signals is given by :

$$\begin{aligned} e(t) &= e_{\mathrm{d}}(t) + e_{\mathrm{n}}(t) \\ &= \left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{h}} - \mathbf{i}_l\right]^T \mathbf{x}(t-\Delta) + \underline{\mathbf{h}}^T \underline{\mathbf{v}}(t), \end{aligned}$$

where

$$\begin{aligned} e_{\mathrm{d}}(t) &= \left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{h}} - \mathbf{i}_l\right]^T \mathbf{x}(t-\Delta), \\ e_{\mathrm{n}}(t) &= \underline{\mathbf{h}}^T \underline{\mathbf{v}}(t), \end{aligned}$$

are the errors due to the desired signal distortion and residual noise, respectively. Since we assume that the desired signal and noise are uncorrelated, the MSE criterion is reduced to the following expression :

$$\begin{aligned} &J\left(\underline{\mathbf{h}}\right) \\ &= \mathbb{E}\left[e_{\mathrm{d}}^2(t)\right] + \mathbb{E}\left[e_{\mathrm{n}}^2(t)\right] \\ &= \mathbb{E}\left[\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{h}} - \mathbf{i}_l\right]^T \mathbf{x}(t-\Delta)\mathbf{x}^T(t-\Delta)\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{h}} - \mathbf{i}_l\right]\right] + \mathbb{E}\left[\underline{\mathbf{h}}^T \underline{\mathbf{v}}(t)\underline{\mathbf{v}}^T(t)\underline{\mathbf{h}}\right] \\ &= \underline{\mathbf{h}}^T \underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\mathbf{R}_{\mathbf{x}}\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{h}} - 2\underline{\mathbf{h}}^T \underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\mathbf{R}_{\mathbf{x}}\mathbf{i}_l + \mathbf{i}_l^T \mathbf{R}_{\underline{\mathbf{x}}}\mathbf{i}_l + \underline{\mathbf{h}}^T \mathbf{R}_{\underline{\mathbf{v}}}\underline{\mathbf{h}} \\ &= \underline{\mathbf{h}}^T \mathbf{R}_{\underline{\mathbf{y}}}\underline{\mathbf{h}} - 2\underline{\mathbf{h}}^T \underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\mathbf{R}_{\mathbf{x}}\mathbf{i}_l + \sigma_x^2. \end{aligned}$$

■

11.2. *Show that the desired signal distortion index can be expressed as*

$$\upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) = \frac{\left[\underline{\mathbf{G}}^T\left(\cos_{\mathrm{d}}\theta\right)\underline{\mathbf{h}} - \mathbf{i}_l\right]^T \mathbf{R}_{\mathbf{x}}\left[\underline{\mathbf{G}}^T\left(\cos_{\mathrm{d}}\theta\right)\underline{\mathbf{h}} - \mathbf{i}_l\right]}{\sigma_x^2}.$$

Solution

$$\begin{aligned}
\upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) &= \frac{\mathbb{E}\left[e_{\mathrm{d}}^2(t)\right]}{\sigma_x^2} \\
&= \frac{\mathbb{E}\left[\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{h}} - \mathbf{i}_l\right]^T \mathbf{x}(t-\Delta)\mathbf{x}^T(t-\Delta)\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{h}} - \mathbf{i}_l\right]\right]}{\sigma_x^2} \\
&= \frac{\left[\underline{\mathbf{G}}^T\left(\cos_{\mathrm{d}}\theta\right)\underline{\mathbf{h}} - \mathbf{i}_l\right]^T \mathbf{R}_{\mathbf{x}}\left[\underline{\mathbf{G}}^T\left(\cos_{\mathrm{d}}\theta\right)\underline{\mathbf{h}} - \mathbf{i}_l\right]}{\sigma_x^2}.
\end{aligned}$$

■

11.3. *Show that the MSEs are related to the different performance measures by*

$$\begin{aligned}
\frac{J_{\mathrm{d}}\left(\underline{\mathbf{h}}\right)}{J_{\mathrm{n}}\left(\underline{\mathbf{h}}\right)} &= \mathrm{iSNR} \times \xi_{\mathrm{n}}\left(\underline{\mathbf{h}}\right) \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) \\
&= \mathrm{oSNR}\left(\underline{\mathbf{h}}\right) \times \xi_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right).
\end{aligned}$$

Solution

$$\begin{aligned}
J_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) &= \mathbb{E}\left[e_{\mathrm{d}}^2(t)\right] = \sigma_x^2 \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right), \\
J_{\mathrm{n}}\left(\underline{\mathbf{h}}\right) &= \mathbb{E}\left[e_{\mathrm{n}}^2(t)\right] = \underline{\mathbf{h}}^T\mathbf{R}_{\underline{\mathbf{v}}}\underline{\mathbf{h}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{J_{\mathrm{d}}\left(\underline{\mathbf{h}}\right)}{J_{\mathrm{n}}\left(\underline{\mathbf{h}}\right)} &= \sigma_x^2 \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) \times \frac{1}{\underline{\mathbf{h}}^T\mathbf{R}_{\underline{\mathbf{v}}}\underline{\mathbf{h}}} \\
&= \frac{\sigma_x^2}{\sigma_{v_1}^2} \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) \times \frac{\sigma_{v_1}^2}{\underline{\mathbf{h}}^T\mathbf{R}_{\underline{\mathbf{v}}}\underline{\mathbf{h}}} \\
&= \mathrm{iSNR} \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) \times \xi_{\mathrm{n}}\left(\underline{\mathbf{h}}\right) \\
&= \frac{\sigma_x^2}{\underline{\mathbf{h}}^T\mathbf{R}_{\underline{\mathbf{x}}}\underline{\mathbf{h}}} \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) \times \frac{\underline{\mathbf{h}}^T\mathbf{R}_{\underline{\mathbf{x}}}\underline{\mathbf{h}}}{\underline{\mathbf{h}}^T\mathbf{R}_{\underline{\mathbf{v}}}\underline{\mathbf{h}}} \\
&= \xi_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) \times \upsilon_{\mathrm{d}}\left(\underline{\mathbf{h}}\right) \times \mathrm{oSNR}\left(\underline{\mathbf{h}}\right).
\end{aligned}$$

■

11.4. *Show that by maximizing the WNG subject to the distortionless constraint, we obtain the DS beamformer:*

$$\underline{\mathbf{h}}_{\mathrm{DS}}\left(\cos\theta_{\mathrm{d}}\right) = \underline{\mathbf{G}}\left(\cos\theta_{\mathrm{d}}\right)\frac{\mathbf{i}_l}{M}.$$

Solution Maximizing the white noise gain subject to the distortionless constraint is equivalent to solving the following optimization problem :

$$\arg\min_{\underline{\mathbf{h}}} \underline{\mathbf{h}}^T\underline{\mathbf{h}} \text{ s.t. } \underline{\mathbf{h}}^T\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}}) = \mathbf{i}_l^T,$$

where

$$\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}}) = \begin{bmatrix} \underline{\mathbf{G}}_1(\cos\theta_{\mathrm{d}}) \\ \underline{\mathbf{G}}_2(\cos\theta_{\mathrm{d}}) \\ \vdots \\ \underline{\mathbf{G}}_M(\cos\theta_{\mathrm{d}}) \end{bmatrix}_{ML_h\times L}.$$

Note that $\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})$ contains L 1-sparse row vectors which are independent and, hence, orthogonal to each other. Therefore, we obtain that $\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})$ is invertible and a diagonal matrix. Define the Lagrangian :

$$\mathcal{L}\left(\underline{\mathbf{h}}, \boldsymbol{\mu}\right) = \underline{\mathbf{h}}^T\underline{\mathbf{h}} - \left[\underline{\mathbf{h}}^T\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}}) - \mathbf{i}_l^T\right]\boldsymbol{\mu}.$$

Hence,

$$\begin{aligned} \nabla_{\underline{\mathbf{h}}}\mathcal{L} &= \mathbf{0} \\ 2\underline{\mathbf{h}} - \underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\boldsymbol{\mu} &= \mathbf{0} \\ \underline{\mathbf{h}} &= \frac{1}{2}\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\boldsymbol{\mu}, \\ \nabla_{\boldsymbol{\mu}}\mathcal{L} &= \mathbf{0} \\ \underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{h}} - \mathbf{i}_l &= \mathbf{0} \\ \frac{1}{2}\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\boldsymbol{\mu} &= \mathbf{i}_l \\ \boldsymbol{\mu} &= 2\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\right]^{-1}\mathbf{i}_l. \end{aligned}$$

As a result,

$$\underline{\mathbf{h}}_{\mathrm{DS}}\left(\cos\theta_{\mathrm{d}}\right) = \underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\right]^{-1}\mathbf{i}_l.$$

The white noise gain is given by,

$$\begin{aligned} &\mathcal{W}(\underline{\mathbf{h}}_{\mathrm{DS}}\left(\cos\theta_{\mathrm{d}}\right)) \\ &= \frac{1}{\underline{\mathbf{h}}_{\mathrm{DS}}^T\left(\cos\theta_{\mathrm{d}}\right)\underline{\mathbf{h}}_{\mathrm{DS}}\left(\cos\theta_{\mathrm{d}}\right)} \\ &= \frac{1}{\mathbf{i}_l^T\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\right]^{-1}\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\right]^{-1}\mathbf{i}_l} \\ &= \frac{1}{\mathbf{i}_l^T\left[\underline{\mathbf{G}}^T(\cos\theta_{\mathrm{d}})\underline{\mathbf{G}}(\cos\theta_{\mathrm{d}})\right]^{-1}\mathbf{i}_l}. \end{aligned}$$

The column vectors of $\underline{\mathbf{G}}(\cos\theta_{\rm d})$, by definition, contain either ones or zeros and the number of nonzero elements ranges from 1 to M. Therefore, by choosing $\mathbf{i}_l$ to contain a 1 in the position which coincides with the maximal (diagonal) element in $\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{G}}(\cos\theta_{\rm d})$, we will minimize $\mathbf{i}_l^T\left[\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{G}}(\cos\theta_{\rm d})\right]^{-1}\mathbf{i}_l$ and, hence, maximize $\mathcal{W}(\underline{\mathbf{h}}_{\rm DS}(\cos\theta_{\rm d}))$. In conclusion, we have,

$$\left[\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{G}}(\cos\theta_{\rm d})\right]^{-1}\mathbf{i}_l = \frac{1}{M}\mathbf{i}_l$$

$$\underline{\mathbf{h}}_{\rm DS}(\cos\theta_{\rm d}) = \underline{\mathbf{G}}(\cos\theta_{\rm d})\frac{\mathbf{i}_l}{M}.$$

■

11.5. *Show that the WNG of the DS beamformer, $\mathcal{W}\left[\underline{\mathbf{h}}_{\rm DS}(\cos\theta_{\rm d})\right]$, is equal to M.*

Solution We saw that the WNG of the DS beamformer is (11.54) :

$$\mathcal{W}[\underline{\mathbf{h}}_{\rm DS}(\cos\theta_{\rm d})] = \frac{1}{\mathbf{i}_l^T\left[\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{G}}(\cos\theta_{\rm d})\right]^{-1}\mathbf{i}_l}$$

Since the matrix product $\mathbf{G}_m^T(\cos\theta_{\rm d})\mathbf{G}_m(\cos\theta_{\rm d})$ is a diagonal matrix, in which the elements are 0 or 1. Hence, the matrix $\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{G}}(\cos\theta_{\rm d}) = \sum_{m=1}^{M}\mathbf{G}_m^T(\cos\theta_{\rm d})\mathbf{G}_m(\cos\theta_{\rm d})$ is also diagonal with elements between 0 and M. We conclude that the position of of the 1 in $\mathbf{i}_l$ must coincide with the position of the maximum element of the diagonal of $\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{G}}(\cos\theta_{\rm d})$. In this case, we get $\mathcal{W}[\underline{\mathbf{h}}_{\rm DS}(\cos\theta_{\rm d})] = M$.

■

11.6. *Show that the maximum DF beamformer is given by*

$$\underline{\mathbf{h}}_{\max}(\cos\theta_{\rm d}) = \varsigma\underline{\mathbf{t}}_1(\cos\theta_{\rm d}),$$

where $\underline{\mathbf{t}}_1(\cos\theta_{\rm d})$ is the eigenvector corresponding to the maximum eigenvalue of the matrix $\mathbf{\Gamma}_{\rm T,0,\pi}^{-1}\underline{\mathbf{G}}(\cos\theta_{\rm d})\underline{\mathbf{G}}^T(\cos\theta_{\rm d})$ and $\varsigma \neq 0$ is an arbitrary real number.

Solution The directivity factor is defined to be:

$$\mathcal{D}(\underline{\mathbf{h}}) = \frac{\underline{\mathbf{h}}^T\underline{\mathbf{G}}(\cos\theta_{\rm d})\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{h}}}{\underline{\mathbf{h}}^T\mathbf{\Gamma}_{\rm T,0,\pi}\underline{\mathbf{h}}}$$

Now,

$$\mathbf{0} = \nabla_{\underline{\mathbf{h}}}\mathcal{D}$$

$$\mathbf{0} = \frac{2\underline{\mathbf{G}}(\cos\theta_{\rm d})\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{h}} \times \underline{\mathbf{h}}^T\mathbf{\Gamma}_{\rm T,0,\pi}\underline{\mathbf{h}} - 2\mathbf{\Gamma}_{\rm T,0,\pi}\underline{\mathbf{h}} \times \underline{\mathbf{h}}^T\underline{\mathbf{G}}(\cos\theta_{\rm d})\underline{\mathbf{G}}^T(\cos\theta_{\rm d})\underline{\mathbf{h}}}{\left[\underline{\mathbf{h}}^T\mathbf{\Gamma}_{\rm T,0,\pi}\underline{\mathbf{h}}\right]^2}.$$