An Exhaustive Class of Linear Filters

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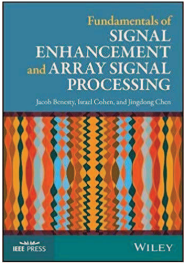

Outline

Introduction

The signal enhancement problems in the time or frequency domain, with one sensor or multiple sensors, are similar.

Within a unified framework, we derive a large class of optimal linear filters as well as filters whose output signal-to-interference-plus-noise ratios (SINRs) are between the conventional maximum SINR and Wiener filters.

We present filters that compromise between interference-plus-noise reduction and desired-signal distortion.

This talk also serves as a bridge between the problem of noise reduction and beamforming.

Signal Model and Problem Formulation

We consider the very general signal model of an observations' signal vector of length M:

$$\begin{aligned}\mathbf{y} &= \begin{bmatrix} y_1 & y_2 & \cdots & y_M \end{bmatrix}^T \\ &= \mathbf{x} + \mathbf{v}_0 + \sum_{n=1}^{N} \mathbf{v}_n \\ &= \mathbf{x} + \mathbf{v}_0 + \mathbf{v}, \end{aligned} \tag{1}$$

where $\mathbf{x}$ is the desired-signal vector, $\mathbf{v}_0$ is the additive white noise signal vector, $\mathbf{v}_n,\ n = 1, 2, \ldots, N$ are N interferences, and $\mathbf{v} = \sum_{n=1}^{N} \mathbf{v}_n$.

All vectors on the right-hand side of (1) are defined similarly to the noisy signal vector, $\mathbf{y}$.

The entries of $\mathbf{y}$ can be, for example, the signals picked up by M sensors.

All signals are considered to be random, complex, circular, zero mean, and stationary.

Furthermore, the vectors $\mathbf{x}$, $\mathbf{v}_0$, and $\mathbf{v}_n, \ n = 1, 2, \ldots, N$ are assumed to be mutually uncorrelated, i.e., $E\left(\mathbf{x}\mathbf{v}_0^H\right) = E\left(\mathbf{x}\mathbf{v}_n^H\right) = E\left(\mathbf{v}_0\mathbf{v}_n^H\right) = E\left(\mathbf{v}_i\mathbf{v}_j^H\right) = \mathbf{0}, \ \forall i \neq j, \ i, j = 1, 2, \ldots, N$.

In this context, the correlation matrix (of size $M \times M$) of the observations can be written as

$$\begin{aligned}\mathbf{\Phi}_{\mathbf{y}} &= E\left(\mathbf{y}\mathbf{y}^H\right) \\ &= \mathbf{\Phi}_{\mathbf{x}} + \mathbf{\Phi}_{\mathbf{v}_0} + \sum_{n=1}^{N} \mathbf{\Phi}_{\mathbf{v}_n} \\ &= \mathbf{\Phi}_{\mathbf{x}} + \mathbf{\Phi}_{\mathbf{v}_0} + \mathbf{\Phi}_{\mathbf{v}} \\ &= \mathbf{\Phi}_{\mathbf{x}} + \mathbf{\Phi}_{\mathrm{in}},\end{aligned} \tag{2}$$

where $\mathbf{\Phi}_{\mathbf{x}} = E\left(\mathbf{x}\mathbf{x}^H\right)$, $\mathbf{\Phi}_{\mathbf{v}_0} = E\left(\mathbf{v}_0\mathbf{v}_0^H\right)$, and $\mathbf{\Phi}_{\mathbf{v}_n} = E\left(\mathbf{v}_n\mathbf{v}_n^H\right)$ are the correlation matrices of $\mathbf{x}$, $\mathbf{v}_0$, and $\mathbf{v}_n$, respectively, $\mathbf{\Phi}_{\mathbf{v}} = \sum_{n=1}^{N} \mathbf{\Phi}_{\mathbf{v}_n}$, and $\mathbf{\Phi}_{\mathrm{in}} = \mathbf{\Phi}_{\mathbf{v}_0} + \mathbf{\Phi}_{\mathbf{v}}$ is the interference-plus-noise correlation matrix.

Since $\mathbf{v}_0$ is assumed to be white, its correlation matrix simplifies to $\mathbf{\Phi}_{\mathbf{v}_0} = \phi_{v_0}\mathbf{I}_M$, where $\phi_{v_0} = E\left(|v_0|^2\right)$ is the variance of v_0, the first component of $\mathbf{v}_0$.

In the rest, the desired-signal and interference correlation matrices are assumed to have the following ranks: $\mathrm{rank}\left(\mathbf{\Phi}_{\mathbf{x}}\right) = R_x \leq M$ and $\mathrm{rank}\left(\mathbf{\Phi}_{\mathbf{v}_n}\right) = R_{v_n} < M$.

Let $R_v = \min\left(\sum_{n=1}^{N} R_{v_n}, M\right)$, we deduce that $\mathrm{rank}\left(\mathbf{\Phi}_{\mathbf{v}}\right) = R_v$ and, obviously, $\mathrm{rank}\left(\mathbf{\Phi}_{\mathrm{in}}\right) = M$.

Then, the objective of signal enhancement (or noise reduction) is to estimate the first element of $\mathbf{x}$, i.e., x_1 (the desired signal sample), from the different second-order statistics available from (2) in the best possible way in some sense.

This should be done in such a way that the noise and the interferences are reduced as much as possible with little or no distortion of the desired signal.

The matrix $\mathbf{\Phi}_{\mathbf{y}}$ can be easily estimated from the observations but $\mathbf{\Phi}_{\mathbf{v}_0}$ and $\mathbf{\Phi}_{\mathbf{v}}$ are more tricky to estimate.

However, in many applications, it is still possible to get reliable estimates of these matrices [1], [2], which will be assumed here.

A very important particular case of the model described above is the conventional beamforming problem, which can be formulated as [3], [4]

$$\mathbf{y} = \mathbf{d}x_1 + \mathbf{v}_0 + \mathbf{v}, \tag{3}$$

where $\mathbf{d}$ is the steering vector of length M, whose first entry is equal to 1.

This vector can be deterministic or random. In the former case, the desired-signal correlation matrix is $\mathbf{\Phi_x} = \phi_{x_1}\mathbf{d}\mathbf{d}^H$, whose rank is, indeed, equal to 1, where $\phi_{x_1} = E\left(|x_1|^2\right)$ is the variance of x_1.

When the steering vector is random, the rank of $\mathbf{\Phi_x}$ is no longer 1 [5].

Some decompositions of the different matrices are necessary in order to fully exploit the structure of the signals.

Using the well-known eigenvalue decomposition [6], the desired-signal correlation matrix can be diagonalized as

$$\mathbf{Q}_\mathbf{x}^H \mathbf{\Phi_x} \mathbf{Q_x} = \mathbf{\Lambda_x}, \tag{4}$$

where

$$\mathbf{Q_x} = \left[\begin{array}{cccc} \mathbf{q}_{\mathbf{x},1} & \mathbf{q}_{\mathbf{x},2} & \cdots & \mathbf{q}_{\mathbf{x},M} \end{array} \right] \tag{5}$$

is a unitary matrix, i.e., $\mathbf{Q}_{\mathbf{x}}^H \mathbf{Q}_{\mathbf{x}} = \mathbf{Q}_{\mathbf{x}} \mathbf{Q}_{\mathbf{x}}^H = \mathbf{I}_M$ and

$$\mathbf{\Lambda}_{\mathbf{x}} = \mathrm{diag}\left(\lambda_{\mathbf{x},1}, \lambda_{\mathbf{x},2}, \ldots, \lambda_{\mathbf{x},M}\right) \tag{6}$$

is a diagonal matrix.

The orthonormal vectors $\mathbf{q}_{\mathbf{x},1}, \mathbf{q}_{\mathbf{x},2}, \ldots, \mathbf{q}_{\mathbf{x},M}$ are the eigenvectors corresponding, respectively, to the eigenvalues $\lambda_{\mathbf{x},1}, \lambda_{\mathbf{x},2}, \ldots, \lambda_{\mathbf{x},M}$ of the matrix $\mathbf{\Phi}_{\mathbf{x}}$, where $\lambda_{\mathbf{x},1} \geq \lambda_{\mathbf{x},2} \geq \cdots \geq \lambda_{\mathbf{x},R_x} > 0$ and $\lambda_{\mathbf{x},R_x+1} = \lambda_{\mathbf{x},R_x+2} = \cdots = \lambda_{\mathbf{x},M} = 0$.

In the same way, the nth interference correlation matrix can be diagonalized as

$$\mathbf{Q}_{\mathbf{v}_n}^H \mathbf{\Phi}_{\mathbf{v}_n} \mathbf{Q}_{\mathbf{v}_n} = \mathbf{\Lambda}_{\mathbf{v}_n}, \tag{7}$$

where the unitary and diagonal matrices $\mathbf{Q}_{\mathbf{v}_n}$ and $\mathbf{\Lambda}_{\mathbf{v}_n}$ are defined in a similar way to $\mathbf{Q}_{\mathbf{x}}$ and $\mathbf{\Lambda}_{\mathbf{x}}$, respectively, with

$\lambda_{\mathbf{v}_n,1} \geq \lambda_{\mathbf{v}_n,2} \geq \cdots \geq \lambda_{\mathbf{v}_n,R_{v_n}} > 0$ and
$\lambda_{\mathbf{v}_n,R_{v_n}+1} = \lambda_{\mathbf{v}_n,R_{v_n}+2} = \cdots = \lambda_{\mathbf{v}_n,M} = 0$.

It may also be useful to diagonalize the matrix $\mathbf{\Phi}_{\mathbf{v}}$ as well, i.e.,

$$\mathbf{Q}_{\mathbf{v}}^H \mathbf{\Phi}_{\mathbf{v}} \mathbf{Q}_{\mathbf{v}} = \mathbf{\Lambda}_{\mathbf{v}}, \quad (8)$$

where $\mathbf{Q}_{\mathbf{v}}$ and $\mathbf{\Lambda}_{\mathbf{v}}$ are similarly defined to $\mathbf{Q}_{\mathbf{x}}$ and $\mathbf{\Lambda}_{\mathbf{x}}$, respectively, with $\lambda_{\mathbf{v},1} \geq \lambda_{\mathbf{v},2} \geq \cdots \geq \lambda_{\mathbf{v},R_v} > 0$ and
$\lambda_{\mathbf{v},R_v+1} = \lambda_{\mathbf{v},R_v+2} = \cdots = \lambda_{\mathbf{v},M} = 0$.

Linear Filtering for Signal Enhancement

By far, the most convenient and practical way to perform signal enhancement, i.e., to estimate the desired-signal, x_1, is by applying a linear filter to the observation signal vector, $\mathbf{y}$, as illustrated in Fig. 1.

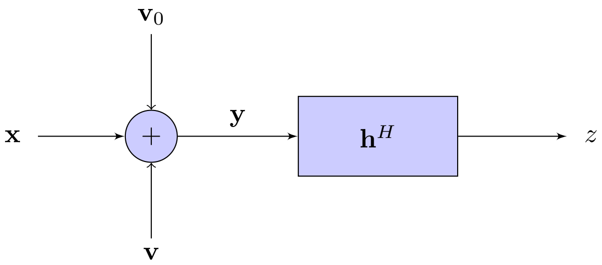

Figure 1: Block diagram of linear filtering.

That is,

$$\begin{aligned} z &= \mathbf{h}^H \mathbf{y} \\ &= \mathbf{h}^H \left(\mathbf{x} + \mathbf{v}_0 + \mathbf{v} \right) \\ &= x_{\text{fd}} + v_{\text{rn}} + v_{\text{ri}}, \end{aligned} \tag{9}$$

where z is the estimate of x_1 or the filter output signal,

$$\mathbf{h} = \left[\begin{array}{cccc} h_1 & h_2 & \cdots & h_M \end{array} \right]^T \tag{10}$$

is a complex-valued filter of length M,

$$x_{\text{fd}} = \mathbf{h}^H \mathbf{x} \tag{11}$$

is the filtered desired signal,

$$v_{\mathrm{rn}} = \mathbf{h}^H \mathbf{v}_0 \tag{12}$$

is the residual noise, and

$$v_{\mathrm{ri}} = \mathbf{h}^H \mathbf{v} \tag{13}$$

is the residual interference.

We deduce that the variance of z is

$$\phi_z = E\left(|z|^2\right) = \phi_{x_{\mathrm{fd}}} + \phi_{v_{\mathrm{rn}}} + \phi_{v_{\mathrm{ri}}},$$

where

$$\phi_{x_{\mathrm{fd}}} = \mathbf{h}^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{h}, \tag{14}$$

$$\phi_{v_{\mathrm{rn}}} = \phi_{v_0} \mathbf{h}^H \mathbf{h}, \tag{15}$$

$$\phi_{v_{\mathrm{ri}}} = \mathbf{h}^H \mathbf{\Phi}_{\mathbf{v}} \mathbf{h}. \tag{16}$$

Performance Measures

Signal-to-Interference-Plus-Noise Ratio

Performance measures are not only useful for the derivation of different kind of optimal filters in some sense but also for their evaluations.

These measures can be divided into two distinct but related categories.

The first category evaluates the noise reduction performance while the second one evaluates the desired-signal distortion.

One of the most fundamental measures in our context is the signal-to-interference-plus-noise ratio (SINR).

The input SINR is a second-order measure, which quantifies the level of the interference-plus-noise present relative to the level of the desired signal.

By taking the first element of $\mathbf{y}$ as the reference, this measure is defined as

$$\mathrm{iSINR} = \frac{\phi_{x_1}}{\phi_{v_0} + \phi_v}, \tag{17}$$

where ϕ_v is the variance of $v = \sum_{n=1}^{N} v_{n1}$, with v_{n1} being the first entry of $\mathbf{v}_n$.

Another interesting measure is the input signal-to-interference ratio (SIR):

$$\mathrm{iSIR} = \frac{\phi_{x_1}}{\phi_v}. \tag{18}$$

The output SINR helps quantify the level of the interference-plus-noise remaining at the filter output signal.

The output SINR is obtained from (14):

$$\begin{aligned} \mathrm{oSINR}\left(\mathbf{h}\right) &= \frac{\phi_{x_{\mathrm{fd}}}}{\phi_{v_{\mathrm{rn}}} + \phi_{v_{\mathrm{ri}}}} \\ &= \frac{\mathbf{h}^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{h}}{\mathbf{h}^H \mathbf{\Phi}_{\mathrm{in}} \mathbf{h}}. \end{aligned} \tag{19}$$

Basically, (19) is the variance of the first signal (filtered desired) from the right-hand side of (14) over the variance of the two other signals (residual interference-plus-noise).

Since the matrix $\mathbf{\Phi}_{\mathrm{in}}$ in the denominator of (19) is full rank, the output SINR is upper bounded.

The objective of the signal enhancement filter is to make the output SINR greater than the input SINR.

Consequently, the quality of the filter output signal may be enhanced as compared to the noisy signal.

It is straightforward to see that the output SIR is

$$\mathrm{oSIR}\left(\mathbf{h}\right) = \frac{\mathbf{h}^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{h}}{\mathbf{h}^H \mathbf{\Phi}_{\mathbf{v}} \mathbf{h}}. \tag{20}$$

Since the matrix $\mathbf{\Phi}_{\mathbf{v}}$ in the denominator of (20) may not be full rank, the output SIR may not be upper bounded.

For the particular filter of length M:

$$\mathbf{i}_{\mathrm{i}} = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array} \right]^T, \tag{21}$$

we have

$$\mathrm{oSINR}\left(\mathbf{i}_{\mathrm{i}}\right) = \mathrm{iSINR}, \quad (22)$$

$$\mathrm{oSIR}\left(\mathbf{i}_{\mathrm{i}}\right) = \mathrm{iSIR}. \quad (23)$$

With the identity filter, $\mathbf{i}_{\mathrm{i}}$, neither the SINR nor the SIR can be improved.

Desired-Signal Distortion Index

Since the noise and interferences are reduced by the filtering operation, so is, in general, the desired-signal.

This implies distortion that we can measure with the desired-signal distortion index, which is defined as the MSE between the desired signal and the filtered desired signal, normalized by the variance of the desired signal, i.e.,

$$\upsilon(\mathbf{h}) = \frac{E\left(\left|x_{\mathrm{fd}} - x_1\right|^2\right)}{E\left(\left|x_1\right|^2\right)} = \frac{\left(\mathbf{h} - \mathbf{i}_{\mathrm{i}}\right)^H \mathbf{\Phi}_{\mathbf{x}} \left(\mathbf{h} - \mathbf{i}_{\mathrm{i}}\right)}{\phi_{x_1}}. \tag{24}$$

The desired-signal distortion index is close to 0 if there is little distortion and expected to be greater than 0 when distortion occurs.

Mean-Squared Error

Error criteria play a critical role in deriving optimal filters.

The MSE [7] as we already know is, by far, the most practical one.

We define the error signal between the estimated and desired signals as

$$\begin{aligned} e &= z - x_1 \\ &= x_{\mathrm{fd}} + v_{\mathrm{rn}} + v_{\mathrm{ri}} - x_1, \end{aligned} \tag{25}$$

which can be written as the sum of three mutually uncorrelated error signals:

$$e = e_{\mathrm{d}} + e_{\mathrm{n}} + e_{\mathrm{i}}, \tag{26}$$

where

$$\begin{aligned} e_{\mathrm{d}} &= x_{\mathrm{fd}} - x_1 \\ &= (\mathbf{h} - \mathbf{i}_{\mathrm{i}})^H \mathbf{x} \end{aligned} \tag{27}$$

is the desired-signal distortion due to the filter,

$$e_{\mathrm{n}} = v_{\mathrm{rn}} = \mathbf{h}^H \mathbf{v}_0 \tag{28}$$

is the residual noise, and

$$e_{\mathrm{i}} = v_{\mathrm{ri}} = \mathbf{h}^H \mathbf{v} \tag{29}$$

is the residual interference.

Therefore, the MSE criterion is

$$\begin{aligned} J(\mathbf{h}) &= E\left(|e|^2\right) \\ &= \phi_{x_1} + \mathbf{h}^H \boldsymbol{\Phi}_{\mathbf{y}} \mathbf{h} - \mathbf{h}^H \boldsymbol{\Phi}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}} - \mathbf{i}_{\mathrm{i}}^T \boldsymbol{\Phi}_{\mathbf{x}} \mathbf{h} \\ &= J_{\mathrm{d}}(\mathbf{h}) + J_{\mathrm{n}}(\mathbf{h}) + J_{\mathrm{i}}(\mathbf{h}), \end{aligned} \tag{30}$$

where

$$\begin{aligned} J_{\mathrm{d}}(\mathbf{h}) &= E\left(|e_{\mathrm{d}}|^2\right) \\ &= (\mathbf{h} - \mathbf{i}_{\mathrm{i}})^H \boldsymbol{\Phi}_{\mathbf{x}} (\mathbf{h} - \mathbf{i}_{\mathrm{i}}) = \phi_{x_1} \upsilon(\mathbf{h}), \end{aligned} \tag{31}$$

$$J_{\mathrm{n}}(\mathbf{h}) = E\left(|e_{\mathrm{n}}|^2\right) = \phi_{v_0} \mathbf{h}^H \mathbf{h}, \tag{32}$$

$$J_{\mathrm{i}}(\mathbf{h}) = E\left(|e_{\mathrm{i}}|^2\right) = \mathbf{h}^H \boldsymbol{\Phi}_{\mathbf{v}} \mathbf{h}. \tag{33}$$

Optimal Filters

Wiener

In this section, we derive a large class of well-known optimal linear filters by fully exploiting the structure of the signals, which was not really done before.

For that, performance measures of the previous section are of great help as well as the appropriate subspace, depending on what we desire.

The Wiener filter is derived by taking the gradient of the MSE, $J(\mathbf{h})$ [eq. (30)], with respect to $\mathbf{h}$ and equating the result to zero:

$$\mathbf{h}_{\mathrm{W}} = \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}}. \tag{34}$$

This optimal filter can also be expressed as

$$\mathbf{h}_{\mathrm{W}} = \left(\mathbf{I}_M - \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{\Phi}_{\mathrm{in}}\right)\mathbf{i}_{\mathrm{i}}. \tag{35}$$

The above formulation is more interesting in practice since it depends on the second-order statistics of the observation and interference-plus-noise signals.

The correlation matrix $\mathbf{\Phi}_{\mathbf{y}}$ can be immediately estimated from the observation signal while the other correlation matrix, $\mathbf{\Phi}_{\mathrm{in}}$, is often known or can be indirectly estimated.

In speech applications, for example, this matrix can be estimated during silences.

Let

$$\mathbf{Q_x} = \begin{bmatrix} \mathbf{Q}'_\mathbf{x} & \mathbf{Q}''_\mathbf{x} \end{bmatrix}, \tag{36}$$

where the $M \times R_x$ matrix $\mathbf{Q}'_\mathbf{x}$ contains the eigenvectors corresponding to the nonzero eigenvalues of $\mathbf{\Phi_x}$ and the $M \times (M - R_x)$ matrix $\mathbf{Q}''_\mathbf{x}$ contains the eigenvectors corresponding to the null eigenvalues of $\mathbf{\Phi_x}$.

It can be verified that

$$\mathbf{I}_M = \mathbf{Q}'_\mathbf{x}\mathbf{Q}'^H_\mathbf{x} + \mathbf{Q}''_\mathbf{x}\mathbf{Q}''^H_\mathbf{x}. \tag{37}$$

Notice that $\mathbf{Q}'_\mathbf{x}\mathbf{Q}'^H_\mathbf{x}$ and $\mathbf{Q}''_\mathbf{x}\mathbf{Q}''^H_\mathbf{x}$ are two orthogonal projection matrices of rank R_x and $M - R_x$, respectively.

Hence, $\mathbf{Q}'_\mathbf{x}\mathbf{Q}'^H_\mathbf{x}$ is the orthogonal projector onto the desired-signal subspace (where all the energy of the desired signal is concentrated) or the range of $\mathbf{\Phi}_{\mathbf{x}}$ and $\mathbf{Q}_{\mathbf{x}}''\mathbf{Q}_{\mathbf{x}}''^{H}$ is the orthogonal projector onto the null subspace of $\mathbf{\Phi}_{\mathbf{x}}$.

With the eigenvalue decomposition of $\mathbf{\Phi}_{\mathbf{x}}$, the correlation matrix of the observations' signal vector can be written as

$$\mathbf{\Phi}_{\mathbf{y}} = \mathbf{Q}_{\mathbf{x}}'\mathbf{\Lambda}_{\mathbf{x}}'\mathbf{Q}_{\mathbf{x}}'^{H} + \mathbf{\Phi}_{\mathrm{in}}, \tag{38}$$

where

$$\mathbf{\Lambda}_{\mathbf{x}}' = \mathrm{diag}\left(\lambda_{\mathbf{x},1}, \lambda_{\mathbf{x},2}, \ldots, \lambda_{\mathbf{x},R_x}\right) \tag{39}$$

is a diagonal matrix of size $R_x \times R_x$.

Determining the inverse of $\mathbf{\Phi}_{\mathbf{y}}$ from (38) with the Woodbury's identity, we get

$$\mathbf{\Phi}_{\mathbf{y}}^{-1} = \mathbf{\Phi}_{\mathrm{in}}^{-1} - \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'\left(\mathbf{\Lambda}_{\mathbf{x}}'^{-1} + \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1}\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathrm{in}}^{-1}. \tag{40}$$

Substituting (40) into (34), leads to another interesting formulation of the Wiener filter:

$$\mathbf{h}_{\mathrm{W}} = \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{\Lambda}_{\mathbf{x}}'^{-1} + \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}. \tag{41}$$

It can be shown that with the optimal Wiener filter given in (34), the output SINR is always greater than or equal to the input SINR, i.e., $\mathrm{oSINR}\left(\mathbf{h}_{\mathrm{W}}\right) \geq \mathrm{iSINR}$ [8].

Example 1

Consider a ULA of M sensors. Suppose that a desired signal impinges on the ULA from the direction θ_x and that an interference impinges on the ULA from the direction θ_v.

Assume that the desired signal received at the first sensor is a complex harmonic random process:

$$x_1(t) = A \exp\left(\jmath 2\pi f_0 t + \jmath\varphi\right),$$

with fixed amplitude A and frequency f_0, and random phase φ, uniformly distributed on the interval from 0 to 2π.

Assume that the interference received at the first sensor, $v_1(t)$, is a random process with the autocorrelation sequence:

$$E\left[v_1(t)v_1(t')\right] = \alpha^{|t-t'|}, \ -1 < \alpha < 1.$$

In addition, the sensors contain thermal white Gaussian noise, whose correlation matrix is $\mathbf{\Phi}_{\mathbf{v}_0} = \phi_{v_0}\mathbf{I}_M$.

The desired signal needs to be recovered from the noisy observation, $\mathbf{y}(t) = \mathbf{d}x_1(t) + \mathbf{v}_0 + \mathbf{v}$, where $\mathbf{d}$ is the steering vector of the desired signal.

Since the desired source impinges on the ULA from the direction θ_x, we have for $m = 2, \ldots, M$:

$$x_m(t) = x_1\left(t - \tau_{x,m}\right) = e^{-\jmath 2\pi f_0 \tau_{x,m}} x_1\left(t\right),$$

where

$$\tau_{x,m} = \frac{(m-1)d\cos\theta_x}{cT_{\mathrm{s}}}$$

is the relative time delay in samples between the desired signals $x_m(t)$ and $x_1(t)$ received at the mth sensor and the first one, and T_s is the sampling interval.

Hence, $\mathbf{x}(t) = \mathbf{d}x_1(t)$, where

$$\mathbf{d} = \begin{bmatrix} 1 & e^{-\jmath 2\pi f_0 \tau_{x,2}} & e^{-\jmath 2\pi f_0 \tau_{x,3}} & \cdots & e^{-\jmath 2\pi f_0 \tau_{x,M}} \end{bmatrix}^T .$$

Similarly,

$$u_m(t) = u_1\left(t - \tau_{v,m}\right),$$

where

$$\tau_{v,m} = \frac{(m-1)d\cos\theta_v}{cT_\mathrm{s}}$$

is the relative time delay in samples between the interferences received at the mth sensor and the first one.

Assuming that the sampling interval satisfies $T_{\mathrm{s}} = \frac{d}{c}$, we have $\tau_{x,m} = (m-1)\cos\theta_x$ and $\tau_{v,m} = (m-1)\cos\theta_v$.

The desired-signal correlation matrix is $\mathbf{\Phi_x} = \phi_{x_1}\mathbf{d}\mathbf{d}^H$, where $\phi_{x_1} = A^2$.

The elements of the $M \times M$ correlation matrix of the interference are $[\mathbf{\Phi_v}]_{i,j} = \alpha^{|\tau_{v,i} - \tau_{v,j}|}$.

The input SINR is

$$\mathrm{iSINR} = 10\log\frac{A^2}{\phi_{v_0}+1} \quad (\mathrm{dB}).$$

The optimal filter $\mathbf{h}_{\mathrm{W}}$ is obtained from (34).

To demonstrate the performance of the Wiener filter, we choose $f_0 = 0.1$, $\theta_x = 90^\circ$, $\theta_v = 0^\circ$, $\alpha = 0.9$, and $\phi_{v_0} = 0.1$.

Figure 2 shows plots of the gain in SINR, $\mathcal{G}(\mathbf{h}_{\mathrm{W}}) = \mathrm{oSINR}(\mathbf{h}_{\mathrm{W}}) / \mathrm{iSINR}$, the MSE, $J(\mathbf{h}_{\mathrm{W}})$, and the desired-signal distortion index, $\upsilon(\mathbf{h}_{\mathrm{W}})$, as a function of the input SINR for different numbers of sensors, M. The gain in SINR is always positive.

For a given input SINR, as the number of sensors increases, the gain in SINR increases, while the MMSE and the desired-signal distortion index decrease.

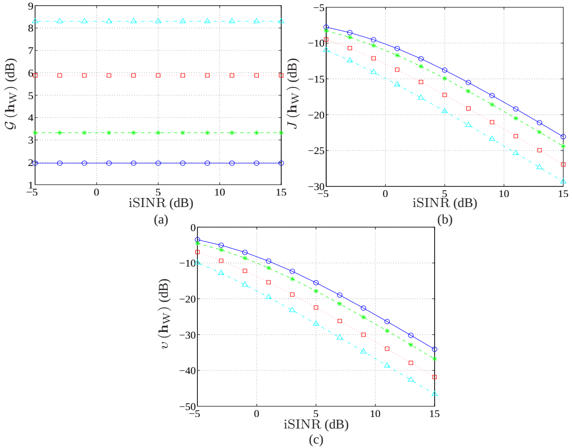

Figure 2: (a) The gain in SINR, (b) the MSE, and (c) the desired-signal distortion index of the Wiener filter for different numbers of sensors, M: $M = 10$ (solid line with circles), $M = 20$ (dashed line with asterisks), $M = 50$ (dotted line with squares), and $M = 100$ (dash-dot line with triangles).

MVDR

In this subsection, we derive a distortionless filter, which is able to reduce the interference-plus-noise, by exploiting the nullspace of $\mathbf{\Phi}_{\mathbf{x}}$.

Using (37), we can write the desired-signal vector as

$$\begin{aligned}\mathbf{x} &= \mathbf{Q}_{\mathbf{x}}\mathbf{Q}_{\mathbf{x}}^{H}\mathbf{x} \\ &= \mathbf{Q}_{\mathbf{x}}'\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{x}.\end{aligned} \tag{42}$$

We deduce from (42) that the distortionless constraint is

$$\mathbf{h}^{H}\mathbf{Q}_{\mathbf{x}}' = \mathbf{i}_{\mathrm{i}}^{T}\mathbf{Q}_{\mathbf{x}}', \tag{43}$$

since, in this case,

$$\begin{aligned}\mathbf{h}^{H}\mathbf{x} &= \mathbf{h}^{H}\mathbf{Q}_{\mathbf{x}}'\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{x} \\ &= \mathbf{i}_{\mathrm{i}}^{T}\mathbf{Q}_{\mathbf{x}}'\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{x} \\ &= x_1.\end{aligned} \tag{44}$$

Now, from the minimization of the criterion:

$$\min_{\mathbf{h}} \left[J_{\mathrm{n}}\left(\mathbf{h}\right) + J_{\mathrm{i}}\left(\mathbf{h}\right) \right] \quad \text{subject to} \quad \mathbf{h}^H \mathbf{Q}'_{\mathbf{x}} = \mathbf{i}_{\mathrm{i}}^T \mathbf{Q}'_{\mathbf{x}}, \tag{45}$$

that is the minimization of the residual interference-plus-noise subject to the distortionless constraint, we find the MVDR filter:

$$\mathbf{h}_{\mathrm{MVDR}} = \mathbf{\Phi}_{\mathrm{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\mathbf{Q}'^H_{\mathbf{x}} \mathbf{\Phi}_{\mathrm{in}}^{-1} \mathbf{Q}'_{\mathbf{x}} \right)^{-1} \mathbf{Q}'^H_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}. \tag{46}$$

It is interesting to compare this filter with the form of the Wiener filter given in (41).

It can be shown that (46) can also be expressed as

$$\mathbf{h}_{\mathrm{MVDR}} = \mathbf{\Phi}_{\mathbf{y}}^{-1} \mathbf{Q}'_{\mathbf{x}} \left(\mathbf{Q}'^H_{\mathbf{x}} \mathbf{\Phi}_{\mathbf{y}}^{-1} \mathbf{Q}'_{\mathbf{x}} \right)^{-1} \mathbf{Q}'^H_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}. \tag{47}$$

It can be verified that, indeed, $J_{\mathrm{d}}\left(\mathbf{h}_{\mathrm{MVDR}}\right) = 0$.

Of course, for $R_x = M$, the MVDR filter simplifies to the identity filter, i.e., $\mathbf{h}_{\mathrm{MVDR}} = \mathbf{i}_{\mathrm{i}}$.

As a consequence, we can state that the higher is the dimension of the nullspace of $\mathbf{\Phi}_{\mathbf{x}}$, the more the MVDR filter is efficient in terms of noise reduction. The best scenario corresponds to $R_x = 1$, which is the form of the MVDR filter that is well known in the literature.

The case $R_x > 1$ was discovered only recently [9], [10].

It can be shown that with the MVDR filter given in (46), the output SINR is always greater than or equal to the input SINR, i.e., $\mathrm{oSINR}(\mathbf{h}_{\mathrm{MVDR}}) \geq \mathrm{iSINR}$ [11].

Example 2

Returning to Example 1, we now employ the MVDR filter, $\mathbf{h}_{\mathrm{MVDR}}$, given in (46).

Figure 3 shows plots of the gain in SINR, $\mathcal{G}\left(\mathbf{h}_{\mathrm{MVDR}}\right)$, and the MSE, $J\left(\mathbf{h}_{\mathrm{MVDR}}\right)$, as a function of the input SINR for different numbers of sensors, M.

The desired-signal distortion index, $\upsilon\left(\mathbf{h}_{\mathrm{MVDR}}\right)$, is zero.

The gain in SINR is always positive. For a given input SINR, as the number of sensors increases, the gain in SINR increases, while the MSE decreases.

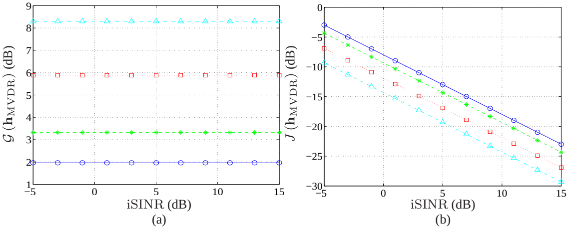

Figure 3: (a) The gain in SINR and (b) the MSE of the MVDR filter for different numbers of sensors, M: $M = 10$ (solid line with circles), $M = 20$ (dashed line with asterisks), $M = 50$ (dotted line with squares), and $M = 100$ (dash-dot line with triangles).

Tradeoff

We are now going to derive a filter that can compromise between interference-plus-noise reduction and desired-signal distortion.

For that, we need to minimize the distortion-based MSE subject to the constraint that the interference-plus-noise reduction-based MSE is equal to some desired value.

Mathematically, this is equivalent to

$$\min_{\mathbf{h}} J_{\mathrm{d}}\left(\mathbf{h}\right) \quad \text{subject to} \quad J_{\mathrm{n}}\left(\mathbf{h}\right) + J_{\mathrm{i}}\left(\mathbf{h}\right) = \aleph\left(\phi_{v_0} + \phi_v\right), \tag{48}$$

where $0 < \aleph < 1$ to ensure that we have some noise reduction.

If we use a Lagrange multiplier, μ, to adjoin the constraint to the cost function, (48) can be rewritten as

$$\mathbf{h}_{\mathrm{T},\mu} = \arg\min_{\mathbf{h}} \mathcal{L}(\mathbf{h}, \mu), \tag{49}$$

with

$$\mathcal{L}(\mathbf{h}, \mu) = J_{\mathrm{d}}\left(\mathbf{h}\right) + \mu\left[J_{\mathrm{n}}\left(\mathbf{h}\right) + J_{\mathrm{i}}\left(\mathbf{h}\right) - \aleph\left(\phi_{v_0} + \phi_v\right)\right] \tag{50}$$

and $\mu > 0$.

From (49), we easily derive the tradeoff filter:

$$\begin{aligned} \mathbf{h}_{\mathrm{T},\mu} &= \left(\mathbf{\Phi}_{\mathbf{x}} + \mu\mathbf{\Phi}_{\mathrm{in}}\right)^{-1}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}} \\ &= \left[\mathbf{\Phi}_{\mathbf{y}} + (\mu-1)\mathbf{\Phi}_{\mathrm{in}}\right]^{-1}\left(\mathbf{\Phi}_{\mathbf{y}} - \mathbf{\Phi}_{\mathrm{in}}\right)\mathbf{i}_{\mathrm{i}}, \end{aligned} \tag{51}$$

where the Lagrange multiplier, μ, satisfies $J_{\mathrm{n}}\left(\mathbf{h}_{\mathrm{T},\mu}\right) + J_{\mathrm{i}}\left(\mathbf{h}_{\mathrm{T},\mu}\right) = \aleph\left(\phi_{v_0} + \phi_v\right)$.

In practice it is not easy to determine the optimal μ.

Therefore, when this parameter is chosen in a heuristic way, we can see that for

- $\mu = 1$, $\mathbf{h}_{\mathrm{T},1} = \mathbf{h}_{\mathrm{W}}$, which is the Wiener filter;
- $\mu = 0$ [if $\mathrm{rank}\left(\mathbf{\Phi}_{\mathbf{x}}\right) = M$], $\mathbf{h}_{\mathrm{T},0} = \mathbf{i}_{\mathrm{i}}$, which is the identity filter;
- $\mu > 1$, results in a filter with low residual interference-plus-noise at the expense of high desired-signal distortion; and
- $\mu < 1$, results in a filter with low desired-signal distortion and small amount of interference-plus-noise reduction.

It can be shown that with the tradeoff filter given in (51), the output SINR is always greater than or equal to the input SINR, i.e., $\mathrm{oSINR}(\mathbf{h}_{\mathrm{T},\mu}) \geq \mathrm{iSINR}, \ \forall \mu \geq 0$ [1].

With the eigenvalue decomposition of $\mathbf{\Phi}_{\mathbf{x}}$ and the Woodbury's identity, we can express the tradeoff filter as

$$\mathbf{h}_{\mathrm{T},\mu} = \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}'_{\mathbf{x}} \left(\mu\mathbf{\Lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}'_{\mathbf{x}}\right)^{-1} \mathbf{Q}'^{H}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}}. \tag{52}$$

This filter is strictly equivalent to the tradeoff filter given in (51), except for $\mu = 0$; indeed, the one in (51) is not defined while the one in (52) leads to the MVDR filter.

Example 3

Returning to Example 1, we now employ the tradeoff filter, $\mathbf{h}_{\mathrm{T},\mu}$, given in (51).

Figures 4 and 5 show plots of the gain in SINR, $\mathcal{G}\left(\mathbf{h}_{\mathrm{T},\mu}\right)$, the MSE, $J\left(\mathbf{h}_{\mathrm{T},\mu}\right)$, and the desired-signal distortion index, $\upsilon\left(\mathbf{h}_{\mathrm{T},\mu}\right)$, as a function of the input SINR for different numbers of sensors, M, for $\mu = 0.5$ and $\mu = 5$, respectively.

The gain in SINR is always positive.

For a given input SINR, as the number of sensors increases, the gain in SINR increases, while the MSE and the desired-signal distortion index decrease.

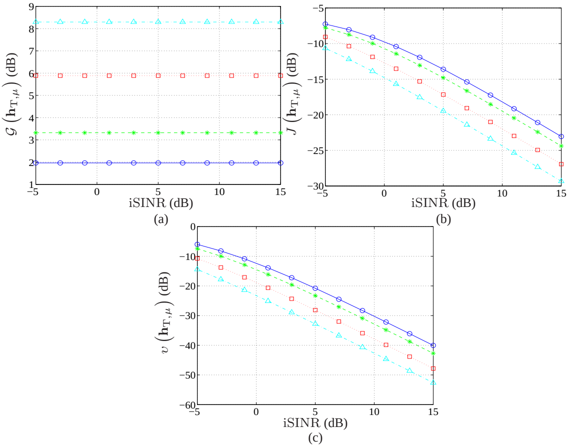

Figure 4: (a) The gain in SINR, (b) the MSE, and (c) the desired-signal distortion index of the tradeoff filter for different numbers of sensors, M, and $\mu = 0.5$: $M = 10$ (solid line with circles), $M = 20$ (dashed line with asterisks), $M = 50$ (dotted line with squares), and $M = 100$ (dash-dot line with triangles).

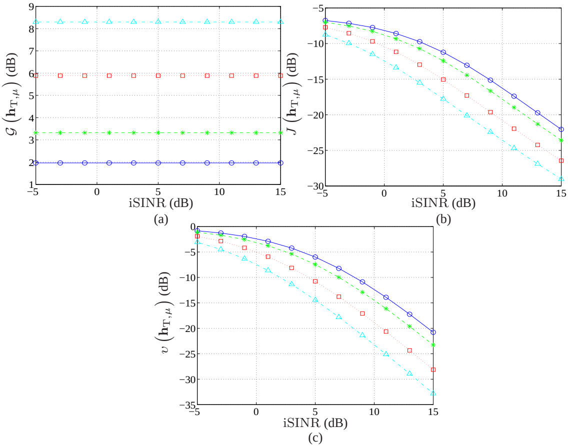

Figure 5: (a) The gain in SINR, (b) the MSE, and (c) the desired-signal distortion index of the tradeoff filter for different numbers of sensors, M, and $\mu = 5$: $M = 10$ (solid line with circles), $M = 20$ (dashed line with asterisks), $M = 50$ (dotted line with squares), and $M = 100$ (dash-dot line with triangles).

LCMV

In this approach, we would like to find a filter that completely cancels one interference, let's say $\mathbf{v}_1$, without any distortion to the desired signal, and attenuates as much as possible the rest of the interference-plus-noise signal.

Let

$$\mathbf{Q}_{\mathbf{v}_1} = \left[\begin{array}{cc} \mathbf{Q}'_{\mathbf{v}_1} & \mathbf{Q}''_{\mathbf{v}_1} \end{array} \right], \tag{53}$$

where the $M \times R_{v_1}$ matrix $\mathbf{Q}'_{\mathbf{v}_1}$ contains the eigenvectors corresponding to the nonzero eigenvalues of $\mathbf{\Phi}_{\mathbf{v}_1}$ and the $M \times (M - R_{v_1})$ matrix $\mathbf{Q}''_{\mathbf{v}_1}$ contains the eigenvectors corresponding to the null eigenvalues of $\mathbf{\Phi}_{\mathbf{v}_1}$.

We can write the interference $\mathbf{v}_1$ as

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{Q}_{\mathbf{v}_1} \mathbf{Q}_{\mathbf{v}_1}^H \mathbf{v}_1 \\ &= \mathbf{Q}'_{\mathbf{v}_1} \mathbf{Q}'^H_{\mathbf{v}_1} \mathbf{v}_1. \end{aligned} \tag{54}$$

We deduce that the constraint to cancel this interference is

$$\mathbf{h}^H \mathbf{Q}'_{\mathbf{v}_1} = \mathbf{0}^T, \tag{55}$$

where $\mathbf{0}$ is the zero vector of length R_{v_1}.

Combining this constraint with the distortionless one, we get

$$\begin{aligned} \mathbf{h}^H \mathbf{C}_{\mathbf{x}\mathbf{v}_1} &= \left[\begin{array}{cc} \mathbf{i}_{\mathrm{i}}^T \mathbf{Q}'_{\mathbf{x}} & \mathbf{0}^T \end{array} \right] \\ &= \mathbf{i}_{\mathrm{c}}^H, \end{aligned} \tag{56}$$

where

$$\mathbf{C}_{\mathbf{x}\mathbf{v}_1} = \begin{bmatrix} \mathbf{Q}'_{\mathbf{x}} & \mathbf{Q}'_{\mathbf{v}_1} \end{bmatrix} \tag{57}$$

is the constraint matrix of size $M \times (R_x + R_{v_1})$ and $\mathbf{i}_\mathrm{c}$ is a vector of length $R_x + R_{v_1}$.

Now, the criterion to optimize is

$$\min_{\mathbf{h}} \left[J_\mathrm{n}(\mathbf{h}) + J_\mathrm{i}(\mathbf{h}) \right] \quad \text{subject to} \quad \mathbf{h}^H \mathbf{C}_{\mathbf{x}\mathbf{v}_1} = \mathbf{i}_\mathrm{c}^H, \tag{58}$$

which leads to the celebrated LCMV filter:

$$\mathbf{h}_\mathrm{LCMV} = \mathbf{\Phi}_\mathrm{in}^{-1} \mathbf{C}_{\mathbf{x}\mathbf{v}_1} \left(\mathbf{C}_{\mathbf{x}\mathbf{v}_1}^H \mathbf{\Phi}_\mathrm{in}^{-1} \mathbf{C}_{\mathbf{x}\mathbf{v}_1} \right)^{-1} \mathbf{i}_\mathrm{c}. \tag{59}$$

It is clear from (59) that for this filter to exist, we must have $M \geq R_x + R_{v_1}$.

An equivalent way to express (59) is

$$\mathbf{h}_{\mathrm{LCMV}} = \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{C}_{\mathbf{x}\mathbf{v}_1}\left(\mathbf{C}_{\mathbf{x}\mathbf{v}_1}^H\mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{C}_{\mathbf{x}\mathbf{v}_1}\right)^{-1}\mathbf{i}_{\mathrm{c}}. \tag{60}$$

While with this filter, we can completely cancel the interference $\mathbf{v}_1$, there is no guarantee that the rest of the interference-plus-noise can be attenuated; in fact, it can even be amplified.

This depends on how M is larger than $R_x + R_{v_1}$. As the difference of these two integers increases, so is the attenuation of the rest of the interference-plus-noise signal.

Example 4

Returning to Example 1, we now assume two uncorrelated complex harmonic random processes as interferences, $\mathbf{v}_1$ and $\mathbf{v}_2$, impinging on the ULA from the directions $\theta_{v_1} = 0°$ and $\theta_{v_2} = 45°$, respectively.

We employ the LCMV filter, $\mathbf{h}_{\mathrm{LCMV}}$, given in (59).

Figure 6 shows plots of the gain in SINR, $\mathcal{G}\left(\mathbf{h}_{\mathrm{LCMV}}\right)$, the MSE, $J\left(\mathbf{h}_{\mathrm{LCMV}}\right)$, and the desired-signal distortion index, $\upsilon\left(\mathbf{h}_{\mathrm{LCMV}}\right)$, as a function of the input SINR for different numbers of sensors, M.

The desired-signal distortion index, $\upsilon\left(\mathbf{h}_{\mathrm{LCMV}}\right)$, is zero.

The gain in SINR is always positive. For a given input SINR, as the number of sensors increases, the gain in SINR increases, while the MSE decreases.

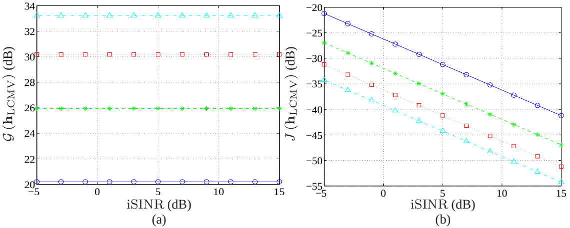

Figure 6: (a) The gain in SINR, (b) the MSE, and (c) the desired-signal distortion index of the LCMV filter for different numbers of sensors, M: $M = 10$ (solid line with circles), $M = 20$ (dashed line with asterisks), $M = 50$ (dotted line with squares), and $M = 100$ (dash-dot line with triangles).

The generalization of this approach to the cancellation of more than one interference is straightforward.

Let's say that we want to cancel the two interferences $\mathbf{v}_1$ and $\mathbf{v}_2$.

First, we take the correlation matrix of the signal $\mathbf{v}_1 + \mathbf{v}_2$.

We perform the eigenvalue decomposition of this matrix as we did for $\mathbf{\Phi}_{\mathbf{v}_1}$.

Then, the derivation of the corresponding LCMV filter is the same as described above.

The only thing that changes is the condition of the filter to exist, which is now $M \geq R_x + R_{v_1} + R_{v_2}$.

Another interesting way to derive the LCMV filter is the following. Let us consider the filters that have the form:

$$\mathbf{h} = \mathbf{Q}''_{\mathbf{v}_1}\mathbf{a}, \tag{61}$$

where $\mathbf{a} \neq \mathbf{0}$ is a shorter filter of length $M - R_{v_1}$. It is easy to observe that

$$\mathbf{h}^H\mathbf{v}_1 = \mathbf{a}^H\mathbf{Q}''^H_{\mathbf{v}_1}\mathbf{v}_1 = \mathbf{0}, \tag{62}$$

since $\mathbf{Q}''^H_{\mathbf{v}_1}\mathbf{Q}'_{\mathbf{v}_1} = \mathbf{0}$. By its nature, the filter $\mathbf{h}$ in (61) cancels the interference.

Substituting (61) into $J_{\mathrm{n}}(\mathbf{h}) + J_{\mathrm{i}}(\mathbf{h})$, we obtain

$$\begin{aligned} J_{\mathrm{n}}(\mathbf{h}) + J_{\mathrm{i}}(\mathbf{h}) &= \phi_{v_0}\mathbf{a}^H\mathbf{a} + \mathbf{a}^H\mathbf{Q}_{\mathbf{v}_1}''^H\mathbf{\Phi}_{\mathbf{v}}\mathbf{Q}_{\mathbf{v}_1}''\mathbf{a} \\ &= \mathbf{a}^H\mathbf{\Phi}_{\mathrm{in}}'\mathbf{a} \\ &= J_{\mathrm{n}}(\mathbf{a}) + J_{\mathrm{i}}(\mathbf{a}), \end{aligned} \tag{63}$$

where

$$\mathbf{\Phi}_{\mathrm{in}}' = \phi_{v_0}\mathbf{I}_{M-R_{v_1}} + \mathbf{Q}_{\mathbf{v}_1}''^H\mathbf{\Phi}_{\mathbf{v}}\mathbf{Q}_{\mathbf{v}_1}'', \tag{64}$$

with $\mathbf{I}_{M-R_{v_1}}$ being the $(M-R_{v_1}) \times (M-R_{v_1})$ identity matrix.

Then, from the criterion:

$$\min_{\mathbf{a}} \left[J_{\mathrm{n}}\left(\mathbf{a}\right) + J_{\mathrm{i}}\left(\mathbf{a}\right) \right] \quad \text{subject to} \quad \mathbf{a}^H \mathbf{Q}_{\mathbf{v}_1}^{\prime\prime H} \mathbf{Q}_{\mathbf{x}}^{\prime} = \mathbf{i}_{\mathrm{i}}^T \mathbf{Q}_{\mathbf{x}}^{\prime}, \tag{65}$$

We find that

$$\mathbf{a}_{\mathrm{LCMV}} = \mathbf{\Phi}_{\mathrm{in}}^{\prime -1} \mathbf{Q}_{\mathbf{v}_1}^{\prime\prime H} \mathbf{Q}_{\mathbf{x}}^{\prime} \left(\mathbf{Q}_{\mathbf{x}}^{\prime H} \mathbf{Q}_{\mathbf{v}_1}^{\prime\prime} \mathbf{\Phi}_{\mathrm{in}}^{\prime -1} \mathbf{Q}_{\mathbf{v}_1}^{\prime\prime H} \mathbf{Q}_{\mathbf{x}}^{\prime} \right)^{-1} \mathbf{Q}_{\mathbf{x}}^{\prime H} \mathbf{i}_{\mathrm{i}}. \tag{66}$$

As a result, another formulation of the LCMV filter is

$$\mathbf{h}_{\mathrm{LCMV}} = \mathbf{Q}_{\mathbf{v}_1}^{\prime\prime} \mathbf{\Phi}_{\mathrm{in}}^{\prime -1} \mathbf{Q}_{\mathbf{v}_1}^{\prime\prime H} \mathbf{Q}_{\mathbf{x}}^{\prime} \left(\mathbf{Q}_{\mathbf{x}}^{\prime H} \mathbf{Q}_{\mathbf{v}_1}^{\prime\prime} \mathbf{\Phi}_{\mathrm{in}}^{\prime -1} \mathbf{Q}_{\mathbf{v}_1}^{\prime\prime H} \mathbf{Q}_{\mathbf{x}}^{\prime} \right)^{-1} \mathbf{Q}_{\mathbf{x}}^{\prime H} \mathbf{i}_{\mathrm{i}}. \tag{67}$$

Maximum SINR

The maximum SINR filter is obtained by maximizing the output SINR as given in (19) from which, we recognize the generalized Rayleigh quotient [6].

Since $\mathbf{\Phi}_{\mathrm{in}}$ is full rank, it is well known that this quotient is maximized with the eigenvector corresponding to the maximum eigenvalue of $\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{\Phi}_{\mathbf{x}}$.

Let us denote by λ_1 the maximum eigenvalue of this matrix and by $\mathbf{t}_1$ the corresponding eigenvector.

Therefore, We have

$$\mathbf{h}_{\mathrm{mSINR}} = \varsigma \mathbf{t}_1, \tag{68}$$

where $\varsigma \neq 0$ is an arbitrary complex number.

We deduce that

$$\mathrm{oSINR}\left(\mathbf{h}_{\mathrm{mSINR}}\right) = \lambda_1 . \tag{69}$$

Clearly, we always have

$$\mathrm{oSINR}\left(\mathbf{h}_{\mathrm{mSINR}}\right) \geq \mathrm{iSINR} \tag{70}$$

and

$$\mathrm{oSINR}\left(\mathbf{h}_{\mathrm{mSINR}}\right) \geq \mathrm{oSINR}\left(\mathbf{h}\right), \ \forall \mathbf{h}. \tag{71}$$

Now, we need to determine ς. One possible good way to find this parameter is by minimizing distortion.

Substituting (68) into $J_{\mathrm{d}}(\mathbf{h})$, we get

$$J_{\mathrm{d}}(\mathbf{h}_{\mathrm{mSINR}}) = \phi_{x_1} + \lambda_1 |\varsigma|^2 - \varsigma^* \mathbf{t}_1^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}} - \varsigma \mathbf{i}_{\mathrm{i}}^T \mathbf{\Phi}_{\mathbf{x}} \mathbf{t}_1. \quad (72)$$

The minimization of the previous expression with respect to ς^* gives

$$\varsigma = \frac{\mathbf{t}_1^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}}{\lambda_1}. \quad (73)$$

We deduce that the maximum SINR filter with minimum distortion is

$$\begin{aligned} \mathbf{h}_{\mathrm{mSINR}} &= \frac{\mathbf{t}_1 \mathbf{t}_1^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}}{\lambda_1} \\ &= \mathbf{t}_1 \mathbf{t}_1^H \mathbf{\Phi}_{\mathrm{in}} \mathbf{i}_{\mathrm{i}}. \end{aligned} \quad (74)$$

Example 5

Returning to Example 1, we now employ the maximum SINR filter, $\mathbf{h}_{\mathrm{mSINR}}$, given in (74).

Figure 7 shows plots of the gain in SINR, $\mathcal{G}\left(\mathbf{h}_{\mathrm{mSINR}}\right)$, and the MSE, $J\left(\mathbf{h}_{\mathrm{mSINR}}\right)$, as a function of the input SINR for different numbers of sensors, M.

The desired-signal distortion index, $\upsilon\left(\mathbf{h}_{\mathrm{mSINR}}\right)$, is zero.

The gain in SINR is always positive.

For a given input SINR, as the number of sensors increases, the gain in SINR increases, while the MSE decreases.

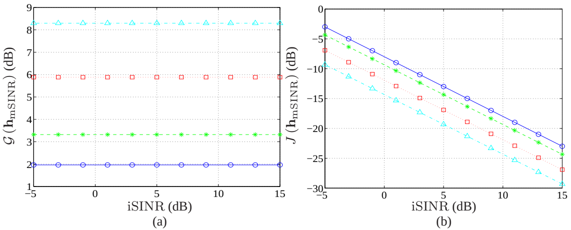

Figure 7: (a) The gain in SINR and (b) the MSE of the maximum SINR filter for different numbers of sensors, M: $M = 10$ (solid line with circles), $M = 20$ (dashed line with asterisks), $M = 50$ (dotted line with squares), and $M = 100$ (dash-dot line with triangles).

Maximum SIR

In the denominator of the output SIR appears the matrix $\mathbf{\Phi_v}$, which can be either full rank or rank deficient.

In the first case, it is easy to derive the maximum SIR filter, which is the eigenvector corresponding to the maximum eigenvalue of $\mathbf{\Phi_v^{-1}\Phi_x}$.

Fundamentally, this scenario is equivalent to what was done in the previous subsection for the maximization of the SINR.

Therefore, we are only interested in the second case, where we assume that $\mathrm{rank}\left(\mathbf{\Phi_v}\right) = R_v < M$.

Let

$$\mathbf{Q}_\mathbf{v} = \begin{bmatrix} \mathbf{Q}'_\mathbf{v} & \mathbf{Q}''_\mathbf{v} \end{bmatrix}, \tag{75}$$

where the $M \times R_v$ matrix $\mathbf{Q}'_\mathbf{v}$ contains the eigenvectors corresponding to the nonzero eigenvalues of $\mathbf{\Phi}_\mathbf{v}$ and the $M \times (M - R_v)$ matrix $\mathbf{Q}''_\mathbf{v}$ contains the eigenvectors corresponding to the null eigenvalues of $\mathbf{\Phi}_\mathbf{v}$.

We are interested in the linear filters of the form:

$$\mathbf{h} = \mathbf{Q}''_\mathbf{v}\mathbf{a}, \tag{76}$$

where $\mathbf{a}$ is a vector of length $M - R_v$.

Since $\mathbf{\Phi_v Q''_v} = \mathbf{0}$ and assuming that $\mathbf{\Phi_x Q''_v} \neq \mathbf{0}$, which is reasonable since $\mathbf{\Phi_x}$ and $\mathbf{\Phi_v}$ cannot be diagonalized by the same orthogonal matrix unless at least one of the two signals x_1 and v is white, we have

$$\mathrm{oSIR}\left(\mathbf{h}\right) = \mathrm{oSIR}\left(\mathbf{Q''_v a}\right) = \infty. \tag{77}$$

As a consequence, the estimate of x_1 is

$$\begin{aligned} \widehat{x}_1 &= \mathbf{h}^H \mathbf{y} \\ &= \mathbf{a}^H \mathbf{Q}''^H_{\mathbf{v}} \mathbf{x} + \mathbf{a}^H \mathbf{Q}''^H_{\mathbf{v}} \mathbf{v}_0 + \mathbf{a}^H \mathbf{Q}''^H_{\mathbf{v}} \mathbf{v} \\ &= \mathbf{a}^H \mathbf{Q}''^H_{\mathbf{v}} \mathbf{x} + \mathbf{a}^H \mathbf{Q}''^H_{\mathbf{v}} \mathbf{v}_0. \end{aligned} \tag{78}$$

We observe from the previous expression that this approach completely cancels the interference.

Now, we need to find $\mathbf{a}$. The best way to find this vector is by minimizing the MSE criterion.

Substituting (76) into (30), we get

$$J(\mathbf{a}) = \phi_{x_1} + \mathbf{a}^H \mathbf{Q}_{\mathbf{v}}''^H \mathbf{\Phi}_{\mathbf{y}} \mathbf{Q}_{\mathbf{v}}'' \mathbf{a} - \mathbf{a}^H \mathbf{Q}_{\mathbf{v}}''^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}} - \mathbf{i}_{\mathrm{i}}^T \mathbf{\Phi}_{\mathbf{x}} \mathbf{Q}_{\mathbf{v}}'' \mathbf{a}. \quad (79)$$

The minimization of the previous expression leads to

$$\mathbf{a}_{\mathrm{mSIR}} = \left(\mathbf{Q}_{\mathbf{v}}''^H \mathbf{\Phi}_{\mathbf{y}} \mathbf{Q}_{\mathbf{v}}'' \right)^{-1} \mathbf{Q}_{\mathbf{v}}''^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}. \quad (80)$$

As a result, the maximum SIR filter with minimum MSE is

$$\mathbf{h}_{\mathrm{mSIR}} = \mathbf{Q}_{\mathbf{v}}'' \left(\mathbf{Q}_{\mathbf{v}}''^H \mathbf{\Phi}_{\mathbf{y}} \mathbf{Q}_{\mathbf{v}}'' \right)^{-1} \mathbf{Q}_{\mathbf{v}}''^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}_{\mathrm{i}}. \quad (81)$$

Example 6

Returning to Example 4, we now employ the maximum SIR filter, $\mathbf{h}_{\mathrm{mSIR}}$, given in (81).

Figure 8 shows plots of the gain in SINR, $\mathcal{G}\left(\mathbf{h}_{\mathrm{mSIR}}\right)$, the MSE, $J\left(\mathbf{h}_{\mathrm{mSIR}}\right)$, and the desired-signal distortion index, $\upsilon\left(\mathbf{h}_{\mathrm{mSIR}}\right)$, as a function of the input SINR for different numbers of sensors, M.

The gain in SINR is always positive.

For a given input SINR, as the number of sensors increases, the gain in SINR increases, while the MSE and the desired-signal distortion index decrease.

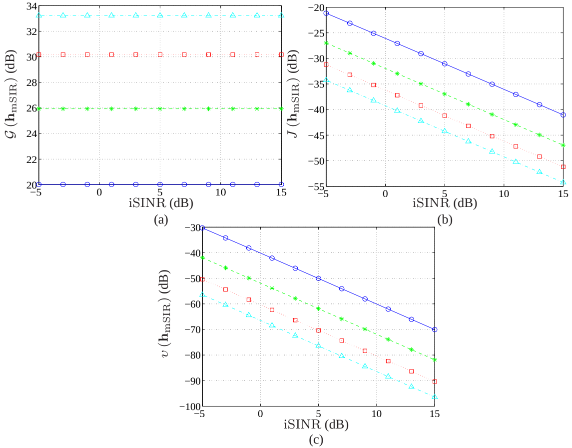

Figure 8: (a) The gain in SINR, (b) the MSE, and (c) the desired-signal distortion index of the maximum SIR filter for different numbers of sensors, M: $M = 10$ (solid line with circles), $M = 20$ (dashed line with asterisks), $M = 50$ (dotted line with squares), and $M = 100$ (dash-dot line with triangles).

All the optimal filters derived in this section are summarized in Table 1.

Table 1: Optimal linear filters for signal enhancement.

Wiener:	$\mathbf{h}_{\mathrm{W}} = \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{\Lambda}_{\mathbf{x}}'^{-1} + \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}$
MVDR:	$\mathbf{h}_{\mathrm{MVDR}} = \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}$
Tradeoff:	$\mathbf{h}_{\mathrm{T},\mu} = \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mu\mathbf{\Lambda}_{\mathbf{x}}'^{-1} + \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}$
LCMV:	$\mathbf{h}_{\mathrm{LCMV}} = \mathbf{Q}_{\mathbf{v}_1}''\mathbf{\Phi}_{\mathrm{in}}'^{-1}\mathbf{Q}_{\mathbf{v}_1}''^{H}\mathbf{Q}_{\mathbf{x}}' \times \left(\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{Q}_{\mathbf{v}_1}''\mathbf{\Phi}_{\mathrm{in}}'^{-1}\mathbf{Q}_{\mathbf{v}_1}''^{H}\mathbf{Q}_{\mathbf{x}}' \right)^{-1} \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}$
Maximum SINR:	$\mathbf{h}_{\mathrm{mSINR}} = \mathbf{t}_1\mathbf{t}_1^{H}\mathbf{\Phi}_{\mathrm{in}}\mathbf{i}_{\mathrm{i}}$
Maximum SIR:	$\mathbf{h}_{\mathrm{mSIR}} = \mathbf{Q}_{\mathbf{v}}'' \left(\mathbf{Q}_{\mathbf{v}}''^{H}\mathbf{\Phi}_{\mathbf{y}}\mathbf{Q}_{\mathbf{v}}'' \right)^{-1} \mathbf{Q}_{\mathbf{v}}''^{H}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}_{\mathrm{i}}$

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