ELIMINATING INTERFERENCE TERMS IN THE WIGNER DISTRIBUTION USING EXTENDED LIBRARIES OF BASES

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ABSTRACT

The Wigner distribution (WD) possesses a number of desirable mathematical properties relevant time-frequency analysis. However, the presence of interference terms renders the WD of multicomponent signals extremely difficult to interpret. In this work, we propose an *adaptive* decomposition of the WD using *extended* libraries of orthonormal bases. A prescribed signal is expanded on a basis of adapted waveforms, that best match the signal components, and subsequently transformed into the Wigner domain. The interference terms are controlled by thresholding the cross WD of interactive basis functions according to their degree of adjacency in an idealized time-frequency plane. This measure is implicitly adapted to the local distribution of the signal, thus compensating for a global nonadaptive threshold. In particular we focus on a shift-invariant decomposition in an *extended library of wavelet packets*. The resulting modified distribution achieves high time-frequency resolution, and is superior in eliminating interference terms associated with bilinear distributions.

1. INTRODUCTION

The Wigner distribution (WD) has long been of special interest, because it possesses a number of desirable mathematical properties [1, 2], including maximal autocomponent concentration in the time-frequency plane. In spite of its desirable properties, the practical application of the WD is often restricted by the presence of interference terms. These often renders the WD of multicomponent signals extremely difficult to interpret.

Several methods, developed to reduce noise and cross-components at the expense of reduced signal concentration, employ some kind of smoothing kernel or windowing [3, 4, 5, 6]. The choice of the kernel dramatically affects the appearance and quality of the resulting time-frequency representation. Hence adaptive representations [5, 7, 8] often exhibit performances far surpassing that of fixed-kernel representations. However, they are either computationally expensive or have a very limited adaptation capability. Another approach striving for cross-term suppression with minimal resolution loss [9, 10, 11] uses the Gabor expansion to decompose the WD. Interference terms are readily identified as cross WD of distinct basis functions. Here, a major drawback is the dependence of the performance on the choice of the Gabor window. An appropriate window selection depends on the data and may differ for different components of the same signal. Furthermore, distinct basis functions which are "close" in the time-frequency plane are often related to a single signal component. Such a cross-term does not necessarily represent an interference term, but rather may have a significant effect on the energy concentration. It is possible to define a distance threshold that discriminates between such auto terms and "true" interference terms. However, this threshold is locally dependent on the distribution and has to be determined adaptively.

In this paper, we propose an *adaptive* decomposition of the WD using *extended* libraries of orthonormal bases. Analogous to the approach in [9, 10, 11], a prescribed signal is expanded on a certain basis and subsequently transformed into the Wigner domain. Here, the basis is selectively constructed out of a redundant library of waveforms to best match the signal components, thereby concentrating its representation into a relatively small number of significant expansion coefficients. The waveforms of the library are well localized in the time-frequency plane, and organized in a binary tree structure facilitating efficient search algorithms for the best basis. In particular we focus on a shift-invariant decomposition in an extended library of wavelet packets [12]. The resultant best-basis representation is preferable to the standard wavelet packet decomposition (WPD) [13] due to its desirable properties. Namely, shift-invariance, lower information cost and improved time-frequency resolution [14].

The interference terms in the Wigner domain are controlled by adaptively thresholding the cross WD of interactive basis functions according to their distance in an idealized time-frequency plane. The distance measure is related to a degree of adjacency by weighing the Euclidean time-frequency distance with the self distribution of the basis-functions. Accordingly, the distance is implicitly adapted to the local distribution of the signal, and local adjustments of a suitable threshold are no longer required.

2. EXTENDED LIBRARY OF WAVELET PACKETS

Overcomplete libraries of waveforms that span redundantly the signal space encourage adaptive signal representations. Instead of representing a prescribed signal in a fixed basis, it is often useful to choose a suitable basis that facilitates a desired application, such as compression, identification, classification or noise removal (denoising). Of particular interest are the libraries of wavelet packet bases, which consist of translations and dilations of wavelet packets, and libraries of local trigonometric bases, comprising sines and cosines multiplied by smooth window functions [13, 15]. The basis functions are localized in the time-frequency plane, and organized in a binary tree structure where efficient search algorithms for the best basis can be implemented.

A serious drawback of the wavelet packet decomposition (WPD) and local cosine decomposition (LCD) [13] is the lack of shift-invariance. Hence we employ modified versions which induce shift-invariance, lower information cost and improved time-frequency resolution [12, 16, 17].

Let us specifically consider the *shift invariant wavelet packet decomposition* (SIWPD) [12, 14]. The library of bases is extended by introducing an additional degree of freedom that adjusts the time-localization of the basis functions. This degree of freedom is practically incorporated into the search algorithm as an adaptive even-odd downsampling. That is, following the low-pass and high-pass filtering, when expanding a parent-node, we retain either all the odd samples or all the even samples, according to the choice which minimizes the cost function.

Let $\{\psi_n(t) : n \in \mathbb{Z}_+\}$ be a wavelet packet family [13] generated by

$$\psi_{2n}(t) = \sqrt{2}\sum_{j\in\mathbb{Z}} h_j \psi_n(2t-j) \tag{1}$$

$$\psi_{2n+1}(t) = \sqrt{2}\sum_{j\in\mathbb{Z}} g_j \psi_n(2t-j) \tag{2}$$

where $g_j = (-1)^j h_{1-j}$, and $\psi_0(t) \equiv \varphi(t)$ is an orthonormal scaling function, satisfying

$$\langle \varphi(t-p), \varphi(t-q)\rangle = \delta_{p,q}, \quad p, q \in \mathbb{Z}\,. \tag{3}$$

The extended library of wavelet packets is defined as the collection of all the orthonormal bases which are subsets of

$$\left\{ B_{\ell,n,m} \;:\; 0 \le \ell \le L,\, 0 \le n, m < 2^{L-\ell} \right\}, \tag{4}$$

where L denotes the finest resolution level, and

$$B_{\ell,n,m} \equiv \left\{ 2^{\ell/2}\psi_n \left[2^{\ell}(t - 2^{-L}m) - k \right] \;:\; 0 \le k < 2^{\ell} \right\}. \tag{5}$$

This library is larger than the standard library by a square power, but can be still structured into a tree configuration which supports fast search algorithms [12]. The additional parameter m provides the crucial degrees of freedom required for the time-adjustment of the basis functions. When an analyzed signal is translated in time by $\tau = q \cdot 2^{-L}$ $(q \in \mathbb{Z})$, a new best-basis is selected whose elements are also translated by τ compared to the former best-basis. Thus the expansion coefficients remain, and the time-frequency representation is shifted in time by the same period.

Compared with the standard WPD, the SIWPD is determined to be advantageous in three respects: The best basis expansion is shift-invariant, characterized by a lower information cost, and the complexity is controlled at the expense of the information cost down to $O(Nlog_2N)$ [14]. These desirable properties advance signal analysis, compression, identification and classification applications. To illustrate the shift-invariant properties of the SIWPD and its enhanced time-frequency representation compared to the standard WPD, we refer to the expansion of the signal $g(t)$, depicted in Fig. 1. This signal, containing $2^7 = 128$ samples, comprises a short pulse, a tone and a component with nonlinear frequency modulation. Figs. 2 and 3 display the best-basis expansions under the WPD and the SIWPD algorithms, respectively, for the signals $g(t)$ and $g(t-2^{-6})$. The sensitivity of WPD to temporal shifts is obvious, while the best-basis SIWPD representation is indeed shift-invariant and characterized by a lower entropy.

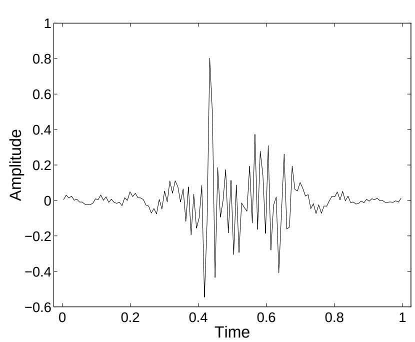


Figure 1. Test signal $g(t)$ consisting of a short pulse, a tone and a nonlinear chirp.

3. ADAPTIVE DECOMPOSITION OF THE WIGNER DISTRIBUTION

The tilings of the time-frequency (TF) plane are idealized representations interconnected with specific basis expansions. A basis-function is symbolized by a rectangular cell whose area is associated with Heisenberg's uncertainty principle, and its shade is proportional to the corresponding coefficient squared. To form time-frequency distributions, we sum up the auto WD of the basis functions and cross WD of pairs which are "close" in the TF plane. Since the cross-term interference is caused by the cross WD of distinct components, one can decide on a distance threshold D in the TF plane, such that farther basis-functions are considered unrelated and their cross WD is discarded. The choice of the best basis maximally concentrates the representation of the signal into a small number of significant expansion coefficients. Thus reducing the computational complexity and decreasing the number of possible cross-terms.

Let $g = \sum_\lambda c_\lambda \varphi_\lambda$ be the best-basis expansion of the signal g. Then its time-frequency distribution is given by

$$\mathrm{TFD}_g = \sum_{\lambda\in\Lambda} |c_\lambda|^2 W_{\varphi_\lambda} + 2\sum_{\{\lambda,\lambda'\}\in\Gamma} \mathrm{Re}\{c_\lambda c^*_{\lambda'} W_{\varphi_\lambda,\varphi_{\lambda'}}\} \tag{6}$$

where W_{φ_λ} is the auto WD of φ_λ and $W_{\varphi_\lambda,\varphi_{\lambda'}}$ is the cross WD of φ_λ and $\varphi_{\lambda'}$:

$$W_{\varphi_\lambda}(t,\omega) = \int \varphi_\lambda(t+\tau/2)\varphi^*_\lambda(t-\tau/2)e^{-j\omega\tau}d\tau \tag{7}$$

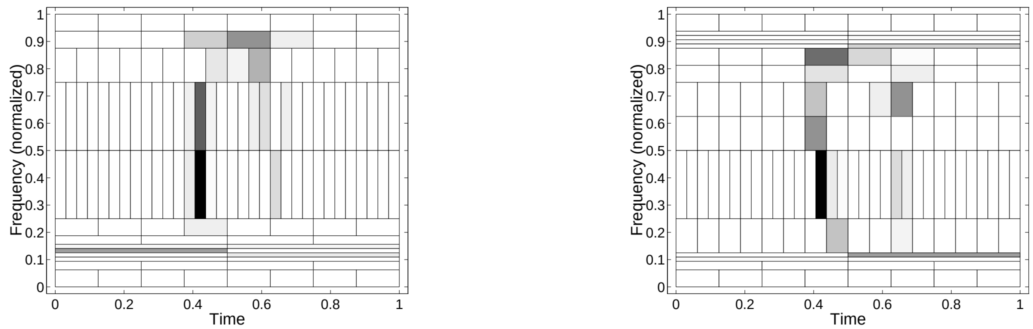
$$W_{\varphi_\lambda,\varphi_{\lambda'}}(t,\omega) = \int \varphi_\lambda(t+\tau/2)\varphi^*_{\lambda'}(t-\tau/2)e^{-j\omega\tau}d\tau \tag{8}$$

The summations in (6) are limited to basis-functions whose coefficients are large enough, and to pairs which are "close" in time-frequency plane. Let δ and D denote respectively thresholds of relative amplitude and time-frequency distance. Then the sets Λ and Γ are defined by

$$\Lambda = \{\lambda \,|\, |c_\lambda| \ge \delta M\}, \quad M \equiv \max_\lambda\{|c_\lambda|\}$$
$$\Gamma = \{\{\lambda,\lambda'\} \,|\, 0 < d(\varphi_\lambda,\varphi_{\lambda'}) \le D,\, |c_\lambda c_{\lambda'}| \ge \delta^2 M^2\}.$$

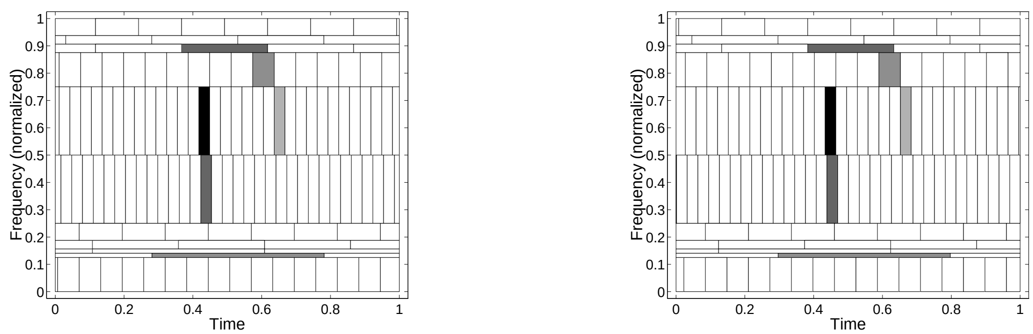
Here, the distance d between a pair of basis-functions is measured by their degree of adjacency:

$$d(\varphi_\lambda,\varphi_{\lambda'}) = \left[\frac{(\bar t_\lambda - \bar t_{\lambda'})^2}{\Delta t_\lambda \Delta t_{\lambda'}} + \frac{(\bar f_\lambda - \bar f_{\lambda'})^2}{\Delta f_\lambda \Delta f_{\lambda'}} \right]^{1/2} \tag{9}$$



(a) (b)

Figure 2. Effect of a temporal shift on the time-frequency representation using the WPD with 8-tap Daubechies wavelet filters: (a) $g(t)$ in its best basis, Entropy= 2.69. (b) $g(t-2^{-6})$ in its best basis, Entropy= 2.72.



(a) (b)

Figure 3. Time-frequency representation using the SIWPD with 8-tap Daubechies wavelet filters: (a) $g(t)$ in its best basis, Entropy= 1.72. (b) $g(t-2^{-6})$ in its best basis, Entropy= 1.72. Compared with the WPD (Fig. 2), beneficial properties are shift-invariance and lower information cost.

where $(\bar{t}_\lambda, \bar{f}_\lambda)$ is the time-frequency position of the basis-function φ_λ, and Δt_λ and Δf_λ are the corresponding time and frequency uncertainties. Accordingly, the Euclidean time-frequency distance between basis-functions is weighed with their self distribution. Since the basis elements are selected to best match the signal's components, their distance is implicitly adapted to the local distribution of the signal, and the distance threshold no longer needs to be dependent on the local distribution. By adjusting the distance threshold D and amplitude threshold δ, one can effectively balance the cross-term interference, the useful properties of the distribution (time/frequency marginals, energy conservation, instantaneous frequency, *etc.* [2]), and the computational complexity.

For the extended library of wavelet packets, introduced in the previous section, the basis-functions are of the form

$$\psi_{\ell,n,m,k}(t) = 2^{\ell/2}\psi_n\left[2^\ell(t-2^{-L}m)-k\right] \qquad (10)$$

where ℓ is the resolution-level index ($0 \le \ell \le L$), n is the frequency index ($0 \le n < 2^{L-\ell}$), m is the shift index ($0 \le m < 2^{L-\ell}$) and k is the position index ($0 \le k < 2^\ell$). Each basis-function is associated with a rectangular tile in the time-frequency plane which is positioned about

$$\bar{t} = 2^{-\ell}k + 2^{-L}m + (2^{L-\ell}-1)C_h + (C_h - C_g)R(n)\,, \qquad (11)$$

$$\bar{f} = 2^{\ell-L}[GC^{-1}(n) + 0.5]\,, \qquad (12)$$

where C_h and C_g are respectively the centers of energy of the low-pass and high-pass quadrature filters h and g [15], defined by

$$C_h = \frac{1}{\|h\|^2}\sum_{j\in\mathbb{Z}} j|h_j|^2\,, \quad C_g = \frac{1}{\|g\|^2}\sum_{j\in\mathbb{Z}} j|g_j|^2\,, \qquad (13)$$

$R(n)$ is an integer obtained by bit reversal of n in a $L-\ell$ bits binary representation, and GC^{-1} is the inverse Gray code permutation. The width and height of the tile are given by

$$\Delta t = 2^{-\ell}\,, \quad \Delta f = 2^{\ell-L}\,. \qquad (14)$$

Fig. 4(a) shows the SIWPD based time-frequency distribution for the signal $g(t)$, attained by utilizing expression (6) combined with the thresholds $D = 1.5$ and $\delta = 0.1$. Fig. 4(b), 4(c), 4(d), 4(e) and 4(f) describe respectively the WD, the Choi-Williams distribution, the spectrogram, the cone-kernel distribution and the reduced interference distribution [6]. Clearly, the SIWPD based time-frequency distribution obtains high resolution and concentration in time-frequency, and is superior in eliminating interference terms associated with the WD.

4. CONCLUSION

Cross terms associated with bilinear distributions are not necessarily interpretable as interference terms. Any signal can be broken up in an infinite number of ways, each of

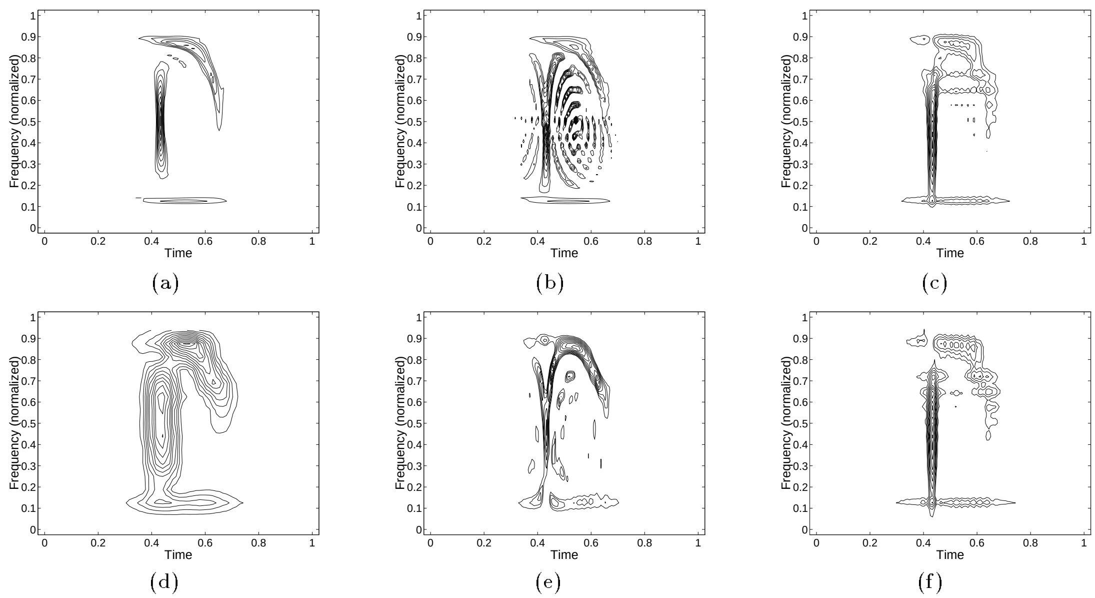


Figure 4. Contour plots for the signal $g(t)$: (a) SIWPD-based time-frequency distribution; (b) Wigner distribution; (c) Choi-Williams distribution; (d) Spectrogram; (e) Cone-kernel distribution; (f) Reduced interference distribution. The SIWPD yields an *adaptive* distribution where high resolution, high concentration, and suppressed interference terms are attainable.

which generates different cross terms. Therefore, it is important to choose an appropriate decomposition that separates the parts which are well delineated in the time-frequency plane. Accordingly, a given signal is expanded into a redundant library of orthonormal bases, from which the best decomposition is selected, and subsequently transformed into the Wigner domain. The discrimination between interference terms and valid cross-terms is determined according to the degree of adjacency and relative amplitudes of the interacting basis functions; Only adjacent pairs whose coefficients are large enough are related to the same component of the signal. The balance between interference terms, concentration and computational complexity is achieved by adjusting the distance and amplitude thresholds.

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