FREQUENCY-BASED DETECTOR FOR IMPROVED RESOLVABILITY OF CLOSELY-SPACED EXPONENTIAL SIGNALS

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ABSTRACT

In this paper, we introduce a new local frequency-based detector for improved resolvability of closely-spaced exponential signals. The frequency resolution problem is defined as a hypothesis-testing problem in the frequency domain, and a corresponding detector, based on the generalized likelihood ratio test is constructed. We derive a theoretical resolution limit in terms of the signal-to-noise ratio, the observable-data length, and the probabilities of detection and false-alarm. Experimental results validate the theoretical results and demonstrate the effectiveness of the proposed detector over a time-based detector.

Index Terms— Detection, resolution, hypothesis testing.

1. INTRODUCTION

Resolvability of exponential signals with nearby frequencies has been studied extensively and is of major importance in diverse fields of signal processing, such as spectral estimation [1–4] and array processing [5]. A wide variety of approaches has been proposed for that purpose, including subspace-based and parametric-estimation methods [5], and model-order selection approaches [1].

Recently, several techniques based on detection theory have been proposed for resolving closely-spaced signals [2, 3]. The resolution problem is formulated as a hypothesis-testing problem, and the small separation between the signals is exploited to derive explicit resolution limits. This formulation allows for defining the resolution in terms of a desired probability of correctly deciding the number of signals, which is extremely essential in real-world applications. In [3], the theoretical resolution limit is derived assuming the signal parameters (e.g., phases, amplitudes and frequencies) are known to the detector. On the other hand, Shahram and Milanfar [2] assume unknown signal parameters and construct a practical detector for *real* closely-spaced sinusoids which is based on a generalized likelihood ratio test (GLRT). Nonetheless, since this detector is locally confined in the time domain, its detection capabilities may be severely worsened when the observed data consists of additional (interfering) spectral components. Moreover, the performance of this detector is insufficient in practical scenarios, where the data may not be entirely available to the detector (e.g., beamspace processing [5]).

In this paper, we extend the time-based detector in [2] and introduce a new local detector operating in the *frequency* domain. The observed signal is first transformed to the spectrum domain using the discrete-Fourier transform (DFT), and a hypothesis-testing problem is formulated in that domain. A corresponding GLRT-based detector is constructed and a theoretical resolution limit is derived as a function of the signal-to-noise ratio (SNR), the observable-data length, and the probabilities of detection and false-alarm. When no interference signal is present, the detectability performance of the proposed detector with only a few samples in the spectrum domain is comparable to that of the time-domain detector. However, a substantial improvement in performance is achieved by the proposed detector when an additional spectral component is present. The theoretical resolution curve is also compared with related theoretical results in the literature and validated by experimental results.

The paper is organized as follows. In Section 2, we formulate the detection problem as a hypothesis-testing problem. In Section 3, we propose a local detector operating in the frequency domain. In Section 4, we analyze the detector performance, and finally in Section 5, we present experimental results which verify the theoretical derivations.

2. PROBLEM FORMULATION

Let an observation signal $y(n)$ consist of two closely-spaced complex exponentials in noise

$$y(n) = \alpha_1 e^{j\theta_1 n} + \alpha_2 e^{j\theta_2 n} + v(n) \triangleq x(n) + v(n) \ , \qquad (1)$$

where α_i $(i = 1, 2)$ are the complex amplitudes of the signals, θ_i $(i = 1, 2)$ are their discrete-time frequencies in radians, $x(n)$ denotes the clean signal, and $v(n)$ is a complex zero-mean white Gaussian noise with variance σ^2. In the sequel, we assume a general uniform sampling of the form $n = \bar{n}, \bar{n}+1, \ldots, \bar{n}+N-1$, with $\bar{n}$ being the first sample index. By "closely-spaced signals" we mean that the spectral resolution of the two exponentials, defined by $|\theta_1 - \theta_2|$, is less than the Fourier resolution limit, defined by

$$\delta_F \triangleq \frac{2\pi}{N} \, . \qquad (2)$$

In such a case, the observation signal $y(n)$ would often provide a single peak in the frequency domain, such that the two frequencies could not be resolved by conventional spectral estimation techniques (e.g., the periodogram). For notational simplicity, we express the

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frequencies of the two exponentials, respectively, as $\theta_1 = \theta_0 - \delta_1$ and $\theta_2 = \theta_0 + \delta_2$, where θ_0 is an "average" frequency and δ_1 and δ_2 are positive quantities whose sum $\delta_1 + \delta_2$ defines the spectral resolution of the two exponentials ($\delta_1 + \delta_2 < \delta_F$). Using the above notations, $x(n)$ from (1) can be rewritten as

$$x(n) = \alpha_1 e^{j(\theta_0 - \delta_1)n} + \alpha_2 e^{j(\theta_0 + \delta_2)n} . \quad (3)$$

Our objective is to decide whether the noisy observation signal $y(n)$ is generated by either two closely-spaced signals or a single-frequency signal. Similarly to [2, 3], we propose to solve the resolution problem by formulating it as a hypothesis-testing problem and to construct a suitable detector. Specifically, the following two hypotheses are considered:

$$\begin{aligned} &H_0 : \delta_1 = 0 \text{ and } \delta_2 = 0 \\ &H_1 : \delta_1 > 0 \text{ or } \delta_2 > 0 \end{aligned} \quad (4)$$

where under hypothesis H_0 a single signal is present and under H_1 two distinct signals are present.

3. PROPOSED DETECTION ALGORITHM

In this section, we derive a local detector in the frequency domain for solving the hypothesis-testing problem (4). We assume that the complex amplitudes α_1 and α_2 and the parameters δ_1 and δ_2 are unknown to the detector and should be estimated from the N data samples. Let us assume for now that the average frequency θ_0 is assumed to be known *a priori* (an efficient estimation procedure for θ_0 is described in Section 5).

For controlling spectral leakage effects, we multiply the observation signal $y(n)$ with a window function $w(n)$ lying in the interval $\bar{n} \le n \le \bar{n} + N - 1$. Applying the discrete-time Fourier transform (DTFT) to the windowed signal $y_w(n) = y(n)w(n)$ yields

$$Y_w(\theta) = \sum_{n=\bar{n}}^{\bar{n}+N-1} y_w(n) e^{-j\theta n} = X_w(\theta) + V_w(\theta) , \quad (5)$$

where

$$X_w(\theta) = \alpha_1 W(\theta - \theta_0 + \delta_1) + \alpha_2 W(\theta - \theta_0 - \delta_2) , \quad (6)$$

and $X_w(\theta)$, $V_w(\theta)$ and $W(\theta)$ are the DTFT of the windowed-version signals $x_w(n)$ and $v_w(n)$, and the window function $w(n)$, respectively. Following a similar reasoning as in [2], we exploit the small frequency separation to approximate the observed signal in the Fourier transform domain using a Taylor expansion around $(\delta_1, \delta_2) = (0, 0)$. Specifically, using the DTFT definition from (5), the second-order Taylor expansion of $X_w(\theta)$ can be expressed as

$$X_w(\theta) \approx \eta_0 W_0(\theta) + \eta_1 W_1(\theta) + \eta_2 W_2(\theta) \quad (7)$$

where

$$W_\ell(\theta) = \frac{1}{\ell!} \sum_{n=\bar{n}}^{\bar{n}+N-1} (jn)^\ell w(n) e^{-j(\theta - \theta_0)n} \ ; \ \ell = 0, 1, 2 \quad (8)$$

and

$$\eta_\ell = (-\delta_1)^\ell \alpha_1 + (\delta_2)^\ell \alpha_2 \ ; \ \ell = 0, 1, 2 . \quad (9)$$

Combining (5) and (7), and evaluating the observable data at equidistance frequencies, we have

$$\bar{Y}_w(k) \approx \sum_{\ell=0}^{2} \eta_\ell \bar{W}_\ell(k) + \bar{V}_w(k) , \quad (10)$$

where $\bar{Y}_w(k) = Y_w(\theta_k)$ for $\theta_k = k \cdot 2\pi / N_1$ ($k = 0, 1, \ldots N_1 - 1$), and $N_1 \ge N$ is the number of samples in the the DFT spectrum [obtained by zero-padding the signal $y_w(n)$]. The signals $\bar{W}_\ell(k)$ and $\bar{V}_w(k)$ are defined similarly. Let k_0 denote the closest discrete frequency bin to θ_0, and let the vector

$$\bar{\mathbf{y}}_w = \left[\ \bar{Y}_w\left(k_0 - \left\lfloor \tfrac{M-1}{2} \right\rfloor\right), \quad \ldots, \quad \bar{Y}_w\left(k_0 + \left\lfloor \tfrac{M}{2} \right\rfloor\right) \ \right]^T \quad (11)$$

represents M samples of the observed DFT signal around k_0. The value of M determines the number of DFT samples that are useful for the detection process, and should be relatively small ($3 \le M \le 7$) to maintain the high SNR around the main-lobe. Finally, equation (10) can be written in a vector form as

$$\bar{\mathbf{y}}_w = \mathbf{W}\boldsymbol{\eta} + \bar{\mathbf{v}}_w \quad (12)$$

where $\mathbf{W} = \left[\ \bar{\mathbf{w}}_0 \quad \bar{\mathbf{w}}_1 \quad \bar{\mathbf{w}}_2 \ \right]$ and $\boldsymbol{\eta} = \left[\ \eta_0 \quad \eta_1 \quad \eta_2 \ \right]^T$ is the model parameters vector, and $\bar{\mathbf{v}}_w$ and $\bar{\mathbf{w}}_\ell$ are defined similarly to $\bar{\mathbf{y}}_w$ as the $M \times 1$ vector representations of $\bar{V}_w(k)$ and $\bar{W}_\ell(k)$, respectively. Using the above notations, the hypothesis-testing problem in (4) can be formulated in a vector form as

$$H_0 : \mathbf{B}\boldsymbol{\eta} = \mathbf{0} \qquad H_1 : \mathbf{B}\boldsymbol{\eta} \ne \mathbf{0} \quad (13)$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \quad (14)$$

Equations (12)-(14) represent the classical linear-model detection problem, which can be solved by the GLRT by substituting the maximum-likelihood (ML) estimate of the parameters vector into the Neyman-Pearson criterion [6]. It is easy to verify from (5) and (10) that the noise in the spectrum domain $\bar{\mathbf{v}}_w$ is a complex zero-mean Gaussian vector with covariance matrix $\mathbf{C_v} = E\left\{\bar{\mathbf{v}}_w \bar{\mathbf{v}}_w^H\right\}$ whose (m, ℓ)th term is

$$\left(\mathbf{C_v}\right)_{m,\ell} = \sigma^2 \sum_{n=\bar{n}}^{\bar{n}+N-1} |w(n)|^2 e^{-j\frac{2\pi}{N_1}(m-\ell)n} . \quad (15)$$

Therefore, given the observable data (12), the ML estimate of the parameters vector $\boldsymbol{\eta}$ is given by

$$\hat{\boldsymbol{\eta}} = \left(\mathbf{W}^H \mathbf{C_v}^{-1} \mathbf{W}\right)^{-1} \mathbf{W}^H \mathbf{C_v}^{-1} \bar{\mathbf{y}}_w . \quad (16)$$

Assuming that σ^2, the variance of the time-series noise signal $v(n)$, is known to the detector, the covariance matrix in the spectrum domain $\mathbf{C_v}$ is also known. Hence, the corresponding GLRT-based detector decides H_1 if [6]

$$T(\bar{\mathbf{y}}_w) = 2\hat{\boldsymbol{\eta}}^H \mathbf{B}^H \left[\mathbf{B}(\mathbf{W}^H \mathbf{C_v}^{-1} \mathbf{W})^{-1} \mathbf{B}^H\right]^{-1} \mathbf{B}\hat{\boldsymbol{\eta}} > \gamma \quad (17)$$

where the threshold γ is chosen as to maintain a constant pre-specified probability of false-alarm[1] .

4. PERFORMANCE ANALYSIS

In this section, we analyze the performance of the proposed detector and derive relations between the attainable resolution and the SNR.

4.1. Relation Between Resolution and SNR

It is well known that the detection performance of the classical linear-model hypothesis test (12)-(14) is [6]

$$P_{FA} = P\left[T\left(\bar{\mathbf{y}}_w\right) > \gamma | H_0\right] = Q_{\chi_4^2}\left(\gamma\right) \tag{18}$$

$$P_D = P\left[T\left(\bar{\mathbf{y}}_w\right) > \gamma | H_1\right] = Q_{\chi_4^2(\lambda)}\left(\gamma\right) \tag{19}$$

where $Q_{\chi_4^2}(\cdot)$ and $Q_{\chi_4^2(\lambda)}(\cdot)$ are the right-tail probabilities for a central and a non-central Chi-Squared PDF, respectively, with 4 degrees of freedom. The noncentrality parameter λ is given by [6]

$$\lambda = 2\boldsymbol{\eta}^H \mathbf{B}^H \left[\mathbf{B}\left(\mathbf{W}^H \mathbf{C}_\mathbf{v}^{-1}\mathbf{W}\right)^{-1}\mathbf{B}^H\right]^{-1}\mathbf{B}\boldsymbol{\eta} \tag{20}$$

where $\boldsymbol{\eta}$ is defined in (12). For pre-specified P_D and P_{FA}, the noncentrality parameter λ is explicitly determined by (18)-(19). Substituting the resulting value of λ, denoted by $\lambda\left(P_{FA}, P_D\right)$, into (20) yields an implicit relation between the achievable resolution $(\delta_1+\delta_2)$ and other relevant model parameters (e.g., window type, observable signal length, noise level, etc.).

To provide further insights into the implicit relation in (20), we consider the interesting case of equal amplitudes, i.e., $\alpha_1 = Ae^{j\phi_1}$ and $\alpha_2 = Ae^{j\phi_2}$, where ϕ_1 and ϕ_2 denotes the initial phases of the two exponentials. Moreover, we assume that θ_0 is properly chosen, such that $\delta_1 \approx \delta_2 = \delta/2$ (see [2, Appendix B]). Accordingly, let the SNR be defined by[2]

$$\text{SNR} \triangleq \frac{A^2}{\sigma^2} \tag{21}$$

and let $\mathbf{D} \triangleq \left[\mathbf{B}\left(\mathbf{W}^H \bar{\mathbf{C}}_\mathbf{v}^{-1}\mathbf{W}\right)^{-1}\mathbf{B}^H\right]^{-1}$, where $\bar{\mathbf{C}}_\mathbf{v} \triangleq \sigma^{-2}\mathbf{C}_\mathbf{v}$. Then, we can rearrange (20) to obtain

$$\text{SNR} = \frac{\lambda\left(P_{FA}, P_D\right)}{\sum_{\ell=1}^{3} d_\ell \cdot (\delta)^{\ell+1}} \triangleq G\left(\delta, P_{FA}, P_D\right), \tag{22}$$

where $d_1 = (1-\cos\Delta\phi)\left|\mathbf{D}_{1,1}\right|^2$, $d_2 = (\cos\Delta\phi)\,\text{Im}\left\{\mathbf{D}_{1,2}\right\}$, $d_3 = 0.25\left(1+\cos\Delta\phi\right)\left|\mathbf{D}_{2,2}\right|^2$, and $\Delta\phi = \phi_2 - \phi_1$ is the initial phase difference. Since the attainable resolution is heavily influenced by the initial phase difference $\Delta\phi$ (see e.g., [3]), it is useful to assume that the phases are uniformly distributed over $[0, 2\pi]$, and to compute the averaged resolution limit as

$$\delta = E_{\Delta\phi}\left\{G^{-1}\left(\text{SNR}, P_{FA}, P_D\right)\right\}, \tag{23}$$

[1] A detector for (13) in the case of an unknown covariance matrix $\mathbf{C}_\mathbf{v}$ can be derived using the Rao test [6].

[2] The SNR definition in (21) is considerably different from that defined in [2] ($\text{SNR} \triangleq \left\|\mathbf{B}\eta\right\|^2/\sigma^2$) due to the influence of the initial phases. Nonetheless, we use the (standard) definition in (21) in order to allow for a fair and simple comparison with common literature results (see Section 4.2).

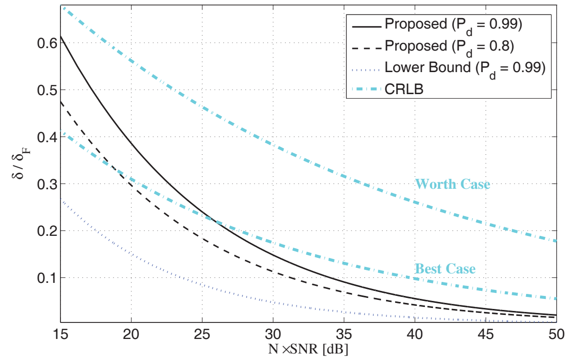

Fig. 1. Comparison of several theoretical (normalized) resolution-limit curves as a function of $N{\cdot}\text{SNR}$.

where the function $G(\cdot)$ is defined in (22). Equations (22)-(23) provide an expression for the (averaged) resolution achievable by the proposed detector as a function of the SNR, P_D and P_{FA}.

4.2. Comparison with Literature Results

In [3], as was previously mentioned, the amplitudes, phases and frequencies of the signals are assumed *known* to the detector. Therefore, the resulting limit curve might be of limited use, since no practical detector, which performs without knowing the exact parameters values, may ever reach this limit. Nonetheless, it is presented here only as a reference for the proposed-detector performance.The resulting (normalized) optimal resolution is approximated by [3]

$$\frac{\delta_\text{O}}{\delta_F} \approx \frac{3.07}{\sqrt{N\cdot\text{SNR}}}\left(\bar{P}_D - 0.5\right), \tag{24}$$

where δ_F is defined in (2). It should be noted that the probability of detection $\bar{P}_D$ in (24) is defined as the probability of correctly deciding the number of signals (either one or two signals), which is different from the definition in (19).

In [4], the resolution limit is defined as twice the root Cramér-Rao (CR) bound on the frequency-estimates variance. Assuming unknown amplitudes, phases, frequencies and noise variance, it was shown that the best resolution is obtained for $\Delta\phi \approx \pi/2$, whereas the worst resolution is obtained for $\Delta\phi \approx 0$. The corresponding CR resolution limits are given by [4]

$$\frac{\delta_\text{CR;best}}{\delta_F} \approx \frac{0.98}{\left(N\cdot\text{SNR}\right)^{1/4}} \tag{25}$$

$$\frac{\delta_\text{CR;worst}}{\delta_F} \approx \frac{1.21}{\left(N\cdot\text{SNR}\right)^{1/6}}. \tag{26}$$

Figure 1 shows the normalized resolution limits as a function of $N{\cdot}\text{SNR}$, obtained by the optimal (reference) bound (24) with $\bar{P}_D = 0.99$, the CR-based limit (25)-(26), and the proposed approach (23) with $P_{FA} = 0.01$ and both $P_D = 0.8$ and 0.99 (for other simulation parameters see Section 5). Clearly, the proposed approach achieves resolution beyond the Fourier limit at reasonable SNR conditions. Moreover, for $N{\cdot}\text{SNR}$ values higher than 25 dB, the resolution achieved by the proposed approach outperforms the best CR-based resolution limit.

5. EXPERIMENTAL RESULTS

In this section, we present experimental results which verify the theoretical results derived in this paper. The data is generated according to (1) and (3) using $|\alpha_1| = |\alpha_2| = A$, $\theta_0 = 0.4\pi$, $N = 51$, and $\bar{n} = -(N-1)/2$. For each value of SNR, we determine the empirical resolution limit by the value of δ whose empirical detection probability equals the desired P_D. We use $P_D = 0.99$ and $P_{FA} = 0.01$. For the proposed frequency-based detector, we employ a rectangular window and utilize five samples in the spectrum domain after zero-padding by a factor of two (i.e., $M = 5$ and $N_1 = 2N$).

For completeness of discussion, we have also evaluated the empirical resolution in a more practical scenario when the average frequency θ_0 is unknown to the detector. In [2], it was suggested to estimate θ_0 using the peak produced by the MUSIC algorithm. To ease the computational burden of such an approach, we propose to estimate θ_0 using a parabolic-fitting procedure by interpolating the discrete spectrum bins. More specifically, let $k_{\max} = \arg\max_k \left|\bar{Y}_w(k)\right|$ [where $\bar{Y}_w(k)$ is defined in (10)], and let $y_0 \triangleq \left|\bar{Y}_w(k_{\max})\right|$, $y_1 \triangleq \left|\bar{Y}_w(k_{\max}+1)\right|$ and $y_{-1} \triangleq \left|\bar{Y}_w(k_{\max}-1)\right|$. Then, an estimate of θ_0 is obtained by the centroid position defined by

$$\hat{\theta}_0 = \frac{2\pi}{N_1}\left(k_{\max} + \frac{y_{-1} - y_1}{2(y_{-1}+y_1) - 4y_0}\right). \qquad (27)$$

Figure 2 shows the resulting resolution curves as a function of $N{\cdot}$SNR, obtained from simulation results and from the theoretical derivations [see (23)]. It can be seen that the experimental results are accurately described by the theoretical resolution curve. Moreover, we observe that the performance of a detector with an estimated θ_0 is comparable to that with a known θ_0. It should be noted that in this case, the detectability performance of a time-domain detector [2] would be similar to that of the proposed frequency-based detector, as both detectors solve an equivalent hypothesis problem.

In the second experiment, we demonstrate the effectiveness of the proposed approach in detecting two closely-spaced signals of the form $Ae^{j(\theta_0-\delta)n} + Ae^{j(\theta_0+\delta)n}$ in the presence of an additive exponential-signal interference $Ae^{j\theta_3 n}$ where $|\theta_0 - \theta_3| > \delta_F$. Note that all exponentials have equal power. We use $N = 101$, $\theta_0 = 0.4\pi$, and $\delta = 0.4\delta_F$, and evaluate the probabilities of detection and false-alarm for an $N{\cdot}$SNR of 30 dB and $\theta_3 = 0.1\pi$ and 0.2π. A comparison is made between the proposed frequency-based detector with rectangular and Hamming windows and the time-based detector proposed in [2], where the latter is slightly modified to the case of complex exponentials. Table 1 specifies the resulting empirical probabilities. We observe that the P_{FA} achieved by the proposed detector is substantially lower than that of the time-based detector, without compromising for lower probability of detection P_D. This is a consequence of efficiently suppressing the interference influence by performing the detection procedure locally in the frequency domain. Therefore, windows with lower sidelobe levels (e.g., Hamming window) yields better detectability performance of the proposed frequency-based approach

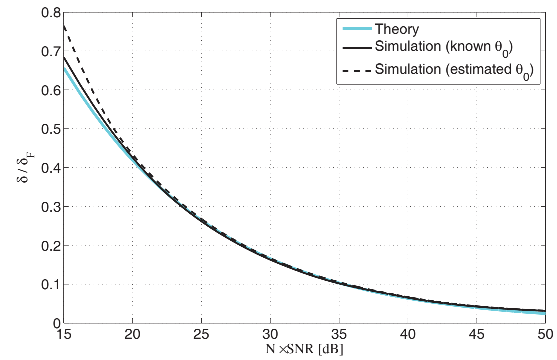

Fig. 2. Comparison of experimental and theoretical and (normalized) resolution curves as a function of $N{\cdot}$SNR.

Table 1. Detectabililty Performances of the Proposed Detector and the Time-Based Detector [2] in the Presence of an Additive Exponential-Signal Interference.

Detector	$\theta_3 = 0.1\pi$		$\theta_3 = 0.2\pi$	
	P_D	P_{FA}	P_D	P_{FA}
Proposed (Rectangular)	0.97	0.05	0.96	0.26
Proposed (Hamming)	**0.99**	**0.02**	**0.99**	**0.09**
Time	0.98	0.11	0.93	0.31

6. CONCLUSIONS

We have introduced a new detection algorithm for resolving complex exponentials with nearby frequencies. Formulating the resolution problem as a classical linear-model detection problem, we employ the GLRT to construct a local frequency-based detector. Theoretical resolution limits were derived and compared to related literature results. The effectiveness of the proposed detector over a time-domain detector was demonstrated in detecting two closely-spaced signals in the presence of an additional interfering spectral component.

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8. REFERENCES

[1] P. Stocia, V. Simonytė, and T. Söderström, "On the resolution performance of spectral analysis," *Signal Process.*, vol. 44, no. 2, pp. 153—161, Jan. 1995.

[2] M. Shahram and P. Milanfar, "On the resolvability of sinusoids with nearby frequencies in the presence of noise," *IEEE Trans. Signal Process.*, vol. 53, no. 7, pp. 2579–2588, Jul. 2005.

[3] A. Amar and A. J. Weiss, "Fundamental limitations on the resolution of deterministic signals," *IEEE Trans. Signal Process.*, vol. 56, no. 11, pp. 5309–5318, Nov. 2008.

[4] E. Dilaveroglu, "Nonmatrix Cramér-Rao bound expressions for high-resolution frequency estimators," *IEEE Trans. Signal Process.*, vol. 46, no. 2, pp. 463–474, Feb. 1998.

[5] H. Krim and M. Viberg, "Two decades of array signal processing research," *IEEE Signal Process. Mag.*, vol. 13, no. 4, pp. 67–94, Jul. 1996.

[6] S. M. Kay, *Fundamentals of Statistical Signal Processing, Detection Theory*. New Jersey: Prentice-Hall, 1998.