ASYMPTOTIC STATIONARITY OF MARKOV-SWITCHING TIME-FREQUENCY GARCH PROCESSES

Ari Abramson and Israel Cohen

Department of Electrical Engineering, Technion - Israel Institute of Technology
Technion City, Haifa 32000, Israel
aari@tx.technion.ac.il, icohen@ee.technion.ac.il

ABSTRACT

Conditions for asymptotic wide-sense stationarity of generalized autoregressive conditional heteroscedasticity (GARCH) processes with regime-switching are necessary for ensuring finite second moments. In this paper, we introduce a stationarity analysis for the Markov-switching time-frequency GARCH (MSTF-GARCH) model which has been recently introduced for modeling nonstationary signals in the time-frequency domain. We obtain a recursive vector form for the unconditional variance by using a representative matrix which is constructed from both the GARCH parameters of each regime, and the regimes' transition probabilities. We show that constraining the spectral radius of that matrix to be less than one is both necessary and sufficient for asymptotic wide-sense stationarity. The generated matrix is also shown to be useful for deriving the asymptotic covariance matrix of the process.

1. INTRODUCTION

Generalized autoregressive conditional heteroscedasticity (GARCH) models with switching-regimes are widely used in the field of econometrics [1–3], and they have recently been utilized for signal processing applications of nonstationary signals such as speech [4, 5]. GARCH processes with Markov-switching regimes, as well as single-regime GARCH processes, are nonstationary as their variances change recursively over time. However, if these processes are asymptotically wide-sense stationary then their second moments are guarantied to be finite. A necessary and sufficient condition for the asymptotic wide-sense stationarity of a (single-regime) GARCH(p, q) process has been developed in [6]. Stationarity conditions for two degenerated cases of GARCH processes with Markov-switching regimes have been derived in [2, 3], and necessary and sufficient conditions for asymptotic wide-sense stationarity have been developed in [7] for the general cases of the GARCH models with Markov-switching regimes of Gray [1], Klaassen [2] and Haas, Mittnik and Paolella [3]. Gray, Klaassen and Haas *et al.*, developed their variants of Markov-switching GARCH models for improved volatility forecasts of financial time-series assuming a noiseless environment, and they all formulated their models in different ways such that the conditional variance does not depend on past active regimes. Therefore, past observations provide complete specifications of current conditional variance for any given regime.

The time-frequency GARCH model with Markov-switching regimes have been recently proposed for modeling nonstationary signals in the time-frequency domain [5]. This Markov-switching time-frequency GARCH model (MSTF-GARCH) naturally extends the time-frequency GARCH model [8, 9] such that the model parameters are regime-dependent. Consequently, the conditional density is a function of the whole process' history, *i.e.,* past observations and active regimes. By using a recursive estimation algorithm, the MSTF-GARCH model has been shown to be useful for modeling nonstationary signals as speech, in the short-time Fourier transform (STFT) domain [5].

In this paper, we analyze the asymptotic stationarity of the MSTF-GARCH model in the general case of m-state Markov chains and (p, q)-order GARCH processes, and give a necessary and sufficient condition for the asymptotic wide-sense stationarity, as well as the asymptotic covariance matrix. We assume no history knowledge of the process except for the model parameters and the stationarity of the Markov chain, and specify the unconditional variance of the process using the expectation of the regime dependent conditional variances. The expectation of the conditional variance at a given regime is recursively constructed from the conditional expectation of both previous conditional and unconditional variances. We define a representative matrix for the model, and show that the unconditional variance converges if and only if the largest absolute eigenvalue of that matrix is less that one. Therefore, this condition is both necessary and sufficient for asymptotic second-order stationarity. The MSTF-GARCH model differs from the three models analyzed in [7] as it is a multivariate, complex-valued model, as well as it naturally formulates the conditional variance by using past active

This research was supported by the Israel Science Foundation (grant no. 1085/05).

regimes and their corresponding conditional variances.

This paper is organized as follows: In Section 2, we review three variants of GARCH models with Markov-switching regimes and define the Markov-switching time-frequency GARCH model. In Section 3, we derive a necessary and sufficient condition for asymptotic wide-sense stationarity of the latter, and give its asymptotic covariance matrix.

2. MARKOV-SWITCHING GARCH MODELS

Three different variants of GARCH models with Markov-switching regimes have been proposed in [1–3]. Each of these models overcomes the problem of dependency on the regime's path encountered when integrating the GARCH model with a regime-switching model.

Let S_t denote the (unobserved) regime at a discrete time t with a realization $s_t \in \{1, ..., m\}$, and assume that $\{S_t\}$ is a hidden Markov chain with transition probabilities $a_{s_t s_{t+1}} \triangleq p\left(S_{t+1} = s_{t+1} \,|\, S_t = s_t\right)$. Gray [1] proposed to model the conditional variance of a Markov-switching GARCH model as dependent on the *expectation* of its past values over the entire set of states, rather than dependent on past states and the corresponding conditional variances. This eliminates the process' dependency on past regimes. Let $\Upsilon^t \triangleq \{\varepsilon_\tau \,|\, \tau \leq t\}$ be the set of observations up to time t and let $\{v_t\}$ be iid random variables with zero-mean and unit-variance. A generalization of Gray's Markov-switching GARCH model to order (p, q) can be formulated as

$$\varepsilon_t = \sigma_{t,s_t} v_t \tag{1}$$

where the state dependent conditional variance follows

$$\begin{aligned}\sigma^2_{t,s_t} &= \xi_{s_t} + \sum_{i=1}^{q} \alpha_{i,s_t} \varepsilon^2_{t-i} + \sum_{j=1}^{p} \beta_{j,s_t} E\left(\varepsilon^2_{t-j} \,|\, \Upsilon^{t-j-1}\right) = \\ &= \xi_{s_t} + \sum_{i=1}^{q} \alpha_{i,s_t} \varepsilon^2_{t-i} \\ &+ \sum_{j=1}^{p} \beta_{j,s_t} \sum_{\tilde{s}=1}^{M} p\left(S_{t-j} = \tilde{s} \,|\, \Upsilon^{t-j-1}\right) \sigma^2_{t-j,\tilde{s}} \, .\end{aligned} \tag{2}$$

This model integrates out the unobserved regime path so the conditional variance can be constructed from previous observations only. As a consequence, there is no path dependency problem although GARCH effects are still allowed.

Klaassen [2] proposed modifying Gray's model by replacing $p\left(S_{t-j} = \tilde{s} \,|\, \Upsilon^{t-j-1}\right)$ in (2) by $p\left(S_{t-j} = \tilde{s} \,|\, \Upsilon^{t-1}, s_t\right)$ while evaluating σ^2_{t,s_t}. Consequently, all available observations are used, as well as the given regime in which the conditional variance is calculated. The conditional variance according to Klaassen's model is given by

$$\begin{aligned}\sigma^2_{t,s_t} &= \xi_{s_t} + \sum_{i=1}^{q} \alpha_{i,s_t} \varepsilon^2_{t-i} \\ &+ \sum_{j=1}^{p} \beta_{j,s_t} \sum_{\tilde{s}=1}^{M} p\left(S_{t-j} = \tilde{s} \,|\, \Upsilon^{t-1}, s_t\right) \sigma^2_{t-j,\tilde{s}} \, .\end{aligned} \tag{3}$$

Another variant of Markov-switching GARCH model has recently been proposed by Haas, Mittnik and Paolella [3]. This model assumes that a Markov chain controls the ARCH parameters at each regime (*i.e.*, ξ_s and $\alpha_{i,s}$), while the *autoregressive* behavior in each regime is subject to the assumption that past conditional variances are in the same regime as that of the current conditional variance. Specifically, the vector of conditional variances $\boldsymbol{\sigma}^2_t \triangleq \left[\sigma^2_{t,1}, ..., \sigma^2_{t,m}\right]^T$ is given by

$$\boldsymbol{\sigma}^2_t = \boldsymbol{\xi} + \sum_{i=1}^{q} \boldsymbol{\alpha}_i \varepsilon^2_{t-i} + \sum_{j=1}^{p} B^{(j)} \boldsymbol{\sigma}^2_{t-j} \, , \tag{4}$$

where $\boldsymbol{\xi} \triangleq [\xi_1, ..., \xi_m]^T$, $\boldsymbol{\alpha}_i \triangleq [\alpha_{i,1}, ..., \alpha_{i,m}]^T$, $i = 1, ..., q$, $\boldsymbol{\beta}_j \triangleq \left[\beta_{j,1}, ..., \beta_{j,m}\right]^T$, $j = 1, ..., p$ and $B^{(j)} \triangleq \text{diag}\left\{\boldsymbol{\beta}_j\right\}$ is a diagonal matrix with elements $\boldsymbol{\beta}_j$ on its diagonal.

Note that the conditional variance in that model at a specific regime depends on previous conditional variances of the same regime through the diagonal matrices $B^{(j)}$. Consequently, this model also allows derivation of the conditional variance at a given regime from past observations only.

A natural extension of the time-frequency GARCH model [8] to Markov-switching has been recently introduced for modeling nonstationary signals in the time-frequency domain [5]. Let $\mathbf{X}_t \in \mathbb{C}^K$ denote a K-dimensional random vector with elements X_{tk}, where $t \geq 0$ is a discrete time index and $k \in \{0, ..., K-1\}$ is the frequency index. Let $\mathcal{X}^t \triangleq \{\mathbf{X}_\tau \,|\, \tau \leq t\}$ represent the data set up to time τ. Let $\mathcal{I}^t \triangleq \{\mathcal{X}^t, \mathcal{S}^t\}$ denote all available information up to time t, which contains the clean signal and the regimes path up to time t, $\mathcal{S}^t \triangleq \{s_\tau \,|\, \tau \leq t\}$. An m-state MSTF-GARCH model of order (p, q) is given by [5]

$$X_{tk} = \sqrt{\lambda_{tk|t-1}} V_{tk} \, , \ \ k = 0, ..., K-1 \, , \tag{5}$$

where $\{V_{tk}\}$ are iid complex random variables with zero-mean, unit-variance and some known probability density. Given the state s_t, the conditional variance of X_{tk}, $\lambda_{tk|t-1,s_t} = E\left\{|X_{tk}|^2 \,|\, \mathcal{I}^{t-1}, s_t\right\}$, is a linear function of previous conditional variances and squared absolute values:

$$\begin{aligned}\lambda_{tk|t-1} \equiv \lambda_{tk|t-1,s_t} &= \xi_{s_t} + \sum_{i=1}^{q} \alpha_{i,s_t} \left|X_{t-i,k}\right|^2 \\ &+ \sum_{j=1}^{p} \beta_{j,s_t} \lambda_{t-j,k|t-j-1} \, .\end{aligned} \tag{6}$$

Sufficient conditions for the positivity of the conditional variances defined by (2), (3), (4) and (6) are

$$\xi_s > 0, \quad \alpha_{i,s} \geq 0, \quad \beta_{j,s} \geq 0, \\ i = 1, ..., q, \quad j = 1, ..., p, \quad s = 1, ..., m . \tag{7}$$

GARCH models with Markov-regimes, are often used for modeling financial time-series. In that case, clean past observations are generally available, and specifying the model without resorting to the hidden regime path is of considerable importance. However, in signal processing applications, the process may be observed in a noisy environment so the process values, as well as the conditional variances and active regimes, are to be estimated. Unlike the above-mentioned GARCH models, the MSTF-GARCH is a multivariate, complex-valued model which naturally extends the GARCH formulation such that the parameters are regime dependent. Accordingly, the model entails the regime path for the construction of the conditional variance from past observations.

3. ASYMPTOTIC STATIONARITY

The conditional variance of Markov-switching GARCH processes changes recursively over time. Consequently, asymptotic wide-sence stationarity is required to ensure a finite second moment [2, 3, 7]. In this section we follow the approach in [7] for deriving a necessary and sufficient condition for the asymptotic wide-sense stationarity and asymptotic covariance matrix of an MSTF-GARCH model.

Assuming a stationary Markov chain with stationary probabilities $\pi_s = p(S_t = s)$, the nonconditional variance vector of an MSTF-GARCH process can be calculated by

$$E\left\{\mathbf{X}_t \odot \mathbf{X}_t^*\right\} = \sum_{s_t} \pi_{s_t} E\left\{\boldsymbol{\lambda}_{t|t-1} \,|\, S_t = s_t\right\} , \tag{8}$$

where $\odot$ denotes a term-by-term multiplication and * denotes complex conjugation. By using the model definitions (5) and (6), we obtain

$$\begin{aligned} E\left\{\boldsymbol{\lambda}_{t|t-1,s_t}\right\} &= E\left\{\boldsymbol{\lambda}_{t|t-1} \,|\, s_t\right\} \\ &= \xi_{s_t}\mathbf{1} + \sum_{i=1}^{q} \alpha_{i,s_t} E\left\{\mathbf{X}_{t-i} \odot \mathbf{X}_{t-i}^* \,|\, s_t\right\} \\ &\quad + \sum_{j=1}^{p} \beta_{j,s_t} E\left\{\boldsymbol{\lambda}_{t-j|t-j-1} \,|\, s_t\right\} , \end{aligned} \tag{9}$$

where $\mathbf{1}$ defines a vector of ones.

Since no prior information is given, we have

$$E\left\{\mathbf{X}_{t-i} \odot \mathbf{X}_{t-i}^* \,|\, s_t\right\} = E\left\{\boldsymbol{\lambda}_{t-i|t-i-1} \,|\, s_t\right\} , \tag{10}$$

and consequently we obtain

$$\begin{aligned} E\left\{\boldsymbol{\lambda}_{t|t-1,s_t}\right\} &= \xi_{s_t}\mathbf{1} \\ &\quad + \sum_{i=1}^{r} \left(\alpha_{i,s_t} + \beta_{i,s_t}\right) E\left\{\boldsymbol{\lambda}_{t-i|t-i-1} \,|\, s_t\right\} , \end{aligned} \tag{11}$$

where $r \triangleq \max\{p, q\}$ and $\alpha_{i,s_t} \triangleq 0 \;\forall i > q$ and $\beta_{i,s_t} \triangleq 0 \;\forall i > p$. The conditional expectation of the conditional variance given a future regime can be obtained using Bayes' rule:

$$E\left\{\boldsymbol{\lambda}_{t-i|t-i-1} \,|\, s_t\right\} = \sum_{s_{t-i}} p\left(s_{t-i} \,|\, s_t\right) E\left\{\boldsymbol{\lambda}_{t-i|t-i-1,s_{t-i}}\right\} , \tag{12}$$

and the conditional state probability in (12) can be evaluated by [7]

$$p\left(s_{t-i} \,|\, s_t\right) = \frac{\pi_{s_{t-i}}}{\pi_{s_t}} \left\{A^i\right\}_{s_{t-i},s_t} , \tag{13}$$

where A is the transition probabilities matrix, *i.e.,* $\{A\}_{ij} \triangleq a_{ij}$.

Let

$$\kappa_{i,s,\tilde{s}} \triangleq \left(\alpha_{i,s} + \beta_{i,s}\right) \frac{\pi_{\tilde{s}}}{\pi_s} \left\{A^i\right\}_{\tilde{s},s} , \tag{14}$$

then by substituting (12) and (13) into (11) we obtain

$$\begin{aligned} E\left\{\boldsymbol{\lambda}_{t|t-1,s_t}\right\} &= \xi_{s_t}\mathbf{1} \\ &\quad + \sum_{i=1}^{r} \sum_{s_{t-i}} \kappa_{i,s_t,s_{t-i}} E\left\{\boldsymbol{\lambda}_{t-i|t-i-1,s_{t-i}}\right\} . \end{aligned} \tag{15}$$

Following the recursive formulation of (15) we define the $m \times m$ matrices $\mathcal{K}_i$, $i = 1, ..., r$ with elements

$$\left\{\mathcal{K}_i\right\}_{s,\tilde{s}} \triangleq \kappa_{i,s,\tilde{s}} , \quad s, \tilde{s} = 1, ..., m , \tag{16}$$

let $\boldsymbol{\lambda}_{tk|t-1,\mathbf{s}_t} \triangleq \left[\lambda_{tk|t-1,S_t=1} , ..., \lambda_{tk|t-1,S_t=m}\right]^T$ be the vector of state dependent, conditional variances at time-frequency bin (t, k). Then by substitution (16) into (15) we obtain for each $k \in \{0, ..., K-1\}$:

$$E\left\{\boldsymbol{\lambda}_{tk|t-1,\mathbf{s}_t}\right\} = \boldsymbol{\xi} + \sum_{i=1}^{r} \mathcal{K}_i E\left\{\boldsymbol{\lambda}_{t-i,k|t-i-1,\mathbf{s}_{t-i}}\right\} . \tag{17}$$

Let $\bar{\boldsymbol{\lambda}}_{tk} \triangleq \left[\boldsymbol{\lambda}_{tk|t-1,\mathbf{s}_t}^T, \boldsymbol{\lambda}_{t-1,k|t-2,\mathbf{s}_{t-1}}^T, ..., \boldsymbol{\lambda}_{t-r+1,k|t-r,\mathbf{s}_{t-r+1}}^T\right]^T$ be a vector of conditional variances at time-frequency bin (t, k), let $\bar{\boldsymbol{\xi}} \triangleq \left[\boldsymbol{\xi}^T, 0, ..., 0\right]^T$ and define an $mr \times mr$ matrix as follows

$$\Psi \triangleq \begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_2 & & \cdots & & \mathcal{K}_r \\ I_m & 0 & & \cdots & & 0 \\ 0 & I_m & & & & \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ & & & & & \\ 0 & \cdots & & 0 & I_m & 0 \end{bmatrix} , \tag{18}$$

where I_m represents the $m \times m$ identity matrix. Then a recursive vector formulation of the expected conditional variance can be written for each $k \in \{0, ..., K-1\}$ as:

$$E\left\{\bar{\boldsymbol{\lambda}}_{tk}\right\} = \bar{\boldsymbol{\xi}} + \Psi E\left\{\bar{\boldsymbol{\lambda}}_{t-1,k}\right\}, \; t > 0\,, \tag{19}$$

with initial condition $\bar{\boldsymbol{\lambda}}_{0,k}$. The solution for the recursive equation (19) is finite for any $t \in \mathbb{N}$ if and only if the largest eigenvalue in modulus of matrix Ψ is less than one [7, Theorem 1], and under this condition

$$\lim_{t\to\infty} E\left\{\bar{\boldsymbol{\lambda}}_{tk}\right\} = (I - \Psi)^{-1}\,\bar{\boldsymbol{\xi}}\,. \tag{20}$$

Define the $m \times m$ matrix Φ by $\{\Phi\}_{ij} = \left\{(I-\Psi)^{-1}\right\}_{ij}$, $i, j = 1, ..., m$, then under stationarity we have

$$\lim_{t\to\infty} E\left\{|X_{tk}|^2\right\} = \boldsymbol{\pi}^T \Phi \boldsymbol{\xi}\,, \tag{21}$$

where $\boldsymbol{\pi}$ denotes the vector of the stationary probabilities of the Markov chain.

Since $\{X_{tk}\}$ are unconditionally zero-mean iid, a necessary and sufficient condition for the asymptotic wide-sense stationarity of the MSTF-GARCH process, defined by (5) and (6), is $\rho(\Psi) < 1$, and the stationary covariance matrix of the process is given by

$$\lim_{t\to\infty} E\left\{\mathbf{X}_t \mathbf{X}_t^H\right\} = \left(\boldsymbol{\pi}^T \Phi \boldsymbol{\xi}\right) I_K\,. \tag{22}$$

This stationarity condition gives a necessary and sufficient condition for the existence of a finite second moment of an MSTF-GARCH process, and it also shows how some regimes (but not all of them) can allow growth of the process' variance over time (*i.e.*, $\sum_i \alpha_{i,s} + \sum_j \beta_{j,s} > 1$ for some states s) and still the process variance will be finite [7].

The stationarity condition of an MSTF-GARCH model is similar to the condition developed in [7] for the model defined by (1) and (3) (which is a generalization of Klaassen's model [2] to m states of order (p, q)). Furthermore, if we look at one frequency index k of the MSTF-GARCH model, its volatility formulation differs from that of Klaassen's model since the conditional variance of the later is a linear function of previous *expectations* of conditional variances rather than their values. However, since these expectations are conditioned on all available information, they equal the expected value of the state dependent *unconditional* variance of an MSTF-GARCH process, given the same state. This explains why the stationarity analysis of the two different models yields similar results.

4. CONCLUSIONS

Conditions for asymptotic wide-sense stationarity of GARCH processes with regime-switching are essential for ensuring the existence of an asymptotic finite second order moment. We have presented stationarity analysis for the Markov-switching time-frequency GARCH model and derived a necessary and sufficient condition for asymptotic second order stationarity. Furthermore, we showed that this condition, as well as the asymptotic variance at any frequency bin index, are similar to those related to Klaassen's model, although the formulation of these two models are different. The discussed model naturally extends a GARCH model in the time-frequency domain to a regime-switching model. Our stationarity analysis is also applicable to any Markov-switching GARCH model which formulates the conditional variance as a linear function of past squared values and conditional variances.

5. REFERENCES

[1] S. F. Gray, "Modeling the conditional distribution of interest rates as a regime-switching process," *Journal of Financial Economics*, vol. 42, pp. 27–62, September 1996.

[2] F. Klaassen, "Improving GARCH volatility forecasts with regime-switching GARCH," *Empirical Economics*, vol. 27, no. 2, pp. 363–394, March 2002.

[3] M. Haas, S. Mittnik, and M. S. Paolella, "A new approach to Markov-switching GARCH models," *Journal of Financial Econometrics*, vol. 2, no. 4, pp. 493–530, Autumn 2004.

[4] M. Abdolahi and H. Amindavar, "GARCH coefficients as feature for speech recognition in persian isolated digit," in *Proc. 30th IEEE Internat. Conf. Acoust. Speech Signal Process., ICASSP-05*, Pennsylvania ,Philadelphia, 17–21 May 2005, pp. I.957–I.960.

[5] A. Abramson and I. Cohen, "Recursive estimation of time-frequency GARCH process with hidden Markov regime," *submitted to IEEE Trans. on Signal Processing*.

[6] T. Bollerslev, "Generalized autoregressive conditional heteroskedasticity," *Journal of Econometrics*, vol. 31, no. 3, pp. 307–327, April 1986.

[7] A. Abramson and I. Cohen, "On the stationarity of GARCH processes with Markov switching regimes," *submitted to Econometric Theory*.

[8] I. Cohen, "From volatility modeling of financial time-series to stochastic modeling and enhancement of speech signals," in *Speech Enhancement*, J. Benesty, S. Makino, and J. Chen, Eds. Springer, 2005, ch. 5, pp. 97–114.

[9] ——, "Modeling speech signals in time-frequency domain using GARCH," *Signal Processing*, vol. 84, no. 12, pp. 2453–2459, December 2004.