PARAMETER ESTIMATION OF HARMONIC LINEAR CHIRPS

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ABSTRACT

We address the problem of estimating the initial frequency and frequency rate of a linear chirp with harmonic components given time samples of the observed signal. As an alternative to the maximum likelihood estimator, which requires an exhaustive search in the initial frequency-frequency rate space, we present a two-step estimation method. First, the signal is separated into its harmonic components. Then, the two parameters of the fundamental component are jointly estimated using a least squares approach given the estimated time-varying phase of each separated component. This method is compared to the maximum likelihood and to a modified high-order ambiguity function based method. Simulations results and a real data example demonstrate the performance of the proposed method. In particular, it is shown that the estimates achieve the Cramer-Rao lower bound at high signal-to-noise ratio and that the two-step method outperforms the high-order ambiguity function based method.

***Index Terms*—** Maximum likelihood estimation, harmonic chirps, Cramer-Rao lower bound.

1. INTRODUCTION

Chirp signals are common in man-made systems such as radar, sonar and communication systems. They can also be found in nature, e.g. bats, whales and dolphins echolocation calls. In addition, chirps can be used to model more complex non-stationary signals such as speech. Harmonics can be used deliberately to increase detectability in active transmission systems such as tissue harmonic imaging [1] or by mammals [2]. They can also occur due to propagation through non-linear media in applications such as speech processing and target localization [3], [4].

Methods for chirps parameter estimation include rank reduction techniques [5], phase unwrapping [6, 7], high order ambiguity function (HAF) based methods [8, 9], multi-linear methods [10], high-order phase function [11], nonlinear least-squares method [12] and subspace methods [13].

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In this work, we consider the case where the signal is modeled as a sum of a known number of harmonic components of a fundamental linear frequency modulated (LFM) chirp (the case where the number of harmonic components is unknown is discussed in [14]). Such a model is usually used to model the chirp signal of bats, whales or birds. We develop a maximum likelihood estimator (MLE) for the two parameters of interest: the initial frequency and the frequency rate of the fundamental LFM. We then present the Cramer-Rao lower bound (CRLB) for the estimation error of the parameters of interest and show that the error of the initial frequency and frequency rate decrease as $1/N^{3/2}$ and $1/N^{5/2}$, respectively, where N is the number of samples.

The MLE requires a high resolution exhaustive search in the initial frequency-frequency rate space and therefore involves a large number of computations. To overcome this burden, we suggest two sub-optimal, low-complexity estimation methods. The first method is a modification of the well-known HAF based estimation method for multi-component polynomial phase signals (PPS) [9]. This method transforms the problem from a two-dimensional maximization to two one-dimensional maximization problems. For a signal with harmonic components, the parameters of interest are estimated by introducing a constraint on the parameters of each component. The second method is the harmonic separate-estimate (Harmonic-SEES) method [14], which is a low-complexity estimator based on the separate-estimate (SEES) approach, used for estimating the coefficients of a constant modulus signals [15]. The proposed method is composed of two steps [14]: The first step is a coarse estimation of the parameters of interest. By using these estimates, the harmonic components are de-chirped and separated by filtering. The process of de-chirping refers to transforming a linear chirp signal to a sinusoid with the same initial frequency. Once the components are separated, a joint least squares approach is used to refine the initial coarse estimates. The HAF based method is presented mainly as a benchmark for the proposed Harmonic-SEES method being very simple and popular. Simulations show that the Harmonic-SEES method achieves the CRLB at high signal to noise ratio (SNR) and outperforms the modified HAF estimation method.

2. PROBLEM FORMULATION

Consider a discrete-time signal composed of M attenuated harmonic components observed in the presence of noise

$$x[n] = \sum_{m=1}^{M} a_m s_m[n;\boldsymbol{\theta}] + v[n],\ n = 0,\ldots,N-1 \quad (1)$$

where $a_m = |a_m|e^{j\mu_m}$ is the unknown complex attenuation of the mth harmonic, and $v[n]$ is a zero mean white Gaussian discrete-time sequence representing the additive noise with a known variance σ_v^2. The mth harmonic component in discrete-time is

$$s_m[n;\boldsymbol{\theta}] = e^{j2\pi m(\theta_1 n + \frac{1}{2}\theta_2 n^2)}, \quad \begin{matrix} n = 0,\ldots,N-1 \\ m = 1,\ldots,M \end{matrix} \quad (2)$$

where $\boldsymbol{\theta} = [\theta_1, \theta_2]^T$. We assume that $\max\{M\theta_1, M(\theta_1 + \theta_2 N)\} \leq 1$ where the first and second arguments correspond to the case of decreasing and increasing chirp harmonics, respectively.

By collecting the N samples of the received signal in (1) we obtain a compact vector-form model given as,

$$\mathbf{x} = \mathbf{S}_\theta \mathbf{a} + \mathbf{v} \quad (3)$$

where we define $\mathbf{x} = [x[0],\ldots,x[N-1]]^T$, $\mathbf{S}_\theta = [\mathbf{s}_1(\boldsymbol{\theta}),\ldots,\mathbf{s}_M(\boldsymbol{\theta})]$, $\mathbf{s}_m(\boldsymbol{\theta}) = [s_m[0;\boldsymbol{\theta}],\ldots,s_m[N-1;\boldsymbol{\theta}]]^T$, $\mathbf{a} = [a_1,\ldots,a_M]^T$, and $\mathbf{v} = [v[0],\ldots,v[N-1]]^T$. The unknown parameter vector of the model, which we would like to estimate, is $\boldsymbol{\psi} = [\boldsymbol{\theta}^T, \mathbf{a}_M^T]^T$.

3. MAXIMUM LIKELIHOOD ESTIMATOR

We now present the MLE of $\boldsymbol{\psi}$. The noise vector, $\mathbf{v}$, is a complex multivariate circularly Gaussian random vector. Therefore, $\mathbf{x} \sim \mathcal{N}_C(\mathbf{S}_\theta\mathbf{a}, \sigma_v^2\mathbf{I}_N)$, where $\mathbf{I}_N$ is the $N \times N$ identity matrix. The negative log likelihood function of $\mathbf{x}$ is given by

$$\ell(\boldsymbol{\psi}) = \frac{1}{2\sigma_v^2}\left\|\mathbf{x} - \mathbf{S}_\theta\mathbf{a}\right\|^2 + \log(K) \quad (4)$$

where K is a constant. The MLE of $\boldsymbol{\psi}$ thus minimizes the first term in (4). By taking the derivative of (4) w.r.t. $\mathbf{a}^H$ and equating to zero, it can be shown that MLE of $\mathbf{a}$ is given by $\hat{\mathbf{a}} = (\mathbf{S}_\theta^H\mathbf{S}_\theta)^{-1}\mathbf{S}_\theta^H\mathbf{x}$. Substituting this estimate in (4) yields the MLE of $\boldsymbol{\theta}$, denoted by $\hat{\boldsymbol{\theta}}^{(\mathrm{MLE})}$,

$$\hat{\boldsymbol{\theta}}^{(\mathrm{MLE})} = \underset{\boldsymbol{\theta}}{\mathrm{argmax}}\ \mathbf{x}^H\mathbf{P}_{\mathbf{S}_\theta}\mathbf{x} \quad (5)$$

where $\mathbf{P}_{\mathbf{S}_\theta} = \mathbf{S}_\theta(\mathbf{S}_\theta^H\mathbf{S}_\theta)^{-1}\mathbf{S}_\theta^H$. Maximizing the term in (5) requires a two-dimensional high resolution search in the initial frequency-frequency rate space.

The covariance matrix of any unbiased estimate of $\boldsymbol{\psi}$, denoted by $\mathrm{cov}(\hat{\boldsymbol{\psi}}) = E[(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi})(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi})^T]$, is lower bounded by the inverse of the Fisher information matrix (FIM), denoted by $\mathbf{J}_{\boldsymbol{\psi},\boldsymbol{\psi}}$. Following [16], it can be shown that the FIM is a block diagonal matrix in the form of

$$\mathbf{J}_{\boldsymbol{\psi},\boldsymbol{\psi}} = \begin{bmatrix} \mathbf{J}_{\boldsymbol{\theta},\boldsymbol{\theta}} & \mathbf{J}_{\boldsymbol{\theta},\mathbf{a}} \\ \mathbf{J}_{\boldsymbol{\theta},\mathbf{a}}^T & \mathbf{J}_{\mathbf{a},\mathbf{a}} \end{bmatrix}. \quad (6)$$

For a large number of samples, i.e. for $N \gg 1$, the CRLB of $\boldsymbol{\theta}$ is approximately given by [14]

$$\mathrm{cov}(\hat{\boldsymbol{\theta}}) \geq \frac{1}{\mathbf{a}^H\mathbf{Q}^2\mathbf{a}}\frac{\sigma_v^2}{\pi^2 N^3}\begin{bmatrix} 24 & -\frac{45}{N} \\ -\frac{45}{N} & \frac{90}{N^2} \end{bmatrix} \quad (7)$$

where $\mathbf{Q} = \mathrm{diag}(1,2,\ldots,M)$. A complete and detailed derivation can be found in [14]. In order to achieve the lower bound, the required search resolution is $1/N^{3/2}$ and $1/N^{5/2}$ for the initial frequency and frequency rate, respectively. Therefore, it can be shown that the computational complexity of the MLE is $\mathcal{O}\left(MN^6\right)$ [14].

4. HIGH ORDER AMBIGUITY FUNCTION

A low complexity sub-optimal parameter estimation method of a multi component PPS based on the HAF was introduced in [9]. The HAF is used to reduce the dimension of the problem to multiple one dimensional problems. We now present how the HAF based parameter estimation can be modified to estimate the parameters of harmonic LFM signals.

The second order ambiguity function is defined as [9]

$$X_2(\theta;\tau) = \sum_{n=0}^{N-1} x_2[n;\tau]e^{-j2\pi\theta n} \quad (8)$$

where $x_2[n;\tau] = x[n]x^*[n-\tau]$ and τ is a delay (measured in samples). It can be shown that applying the second order ambiguity function to an LFM signal yields a complex sinusoid signal with a frequency proportional to the chirp rate [9]. That is, $x_2[n;\tau]$ will be a sum of M complex sinusoids at the frequencies $\theta_m = \tau m\theta_2$. The problem of estimating the frequency rate becomes a discrete time Fourier transform (DTFT) maximization problem. Using the harmonic relation, we estimate the frequency rate as

$$\hat{\theta}_2^{(\mathrm{HAF})} = \frac{1}{\tau}\underset{\theta_2}{\mathrm{argmax}}\sum_{m=1}^{M} X_2(m\theta_2;\tau). \quad (9)$$

Once the frequency rate is estimated, we define the following set of signals

$$x_m[n] = x[n]e^{-j2\pi\frac{1}{2}m\hat{\theta}_2^{(\mathrm{HAF})}n^2}. \quad (10)$$

where $m = 1,\ldots,M$. Each signal $x_m[n]$ is a complex sinusoid in presence of noise and interference from the other components. Again, the initial frequency is estimated using one a dimensional DTFT maximization

$$\hat{\theta}_1^{(\mathrm{HAF})} = \underset{\theta_1}{\operatorname{argmax}} \sum_{m=1}^{M} x_m[n] e^{-j2\pi m\theta_1 n}. \qquad (11)$$

The HAF method relies on DTFT maximization. Therefore, a high resolution search is still required. However, the HAF reduces the problem to two one-dimensional searches. The computational complexity of the HAF based method is $\mathcal{O}\left(N^{7/2}\right)$ [14], which is substantially less than that of the MLE and independent on the number of harmonic components. We also note that the HAF estimation method suffers from error propagation. That is, error in the frequency rate estimate has a great impact on the estimation of the initial frequency.

5. HARMONIC SEPARATE-ESTIMATE METHOD

Both the MLE of $\boldsymbol{\theta}$ in (5), and the HAF estimation method, in (9) and (11), require high resolution search. In order to reduce it, a low complexity method for the estimation of $\boldsymbol{\theta}$ is presented, which we term as an Harmonic-SEES method.

The proposed method consists of two steps. First, the signal is separated to M harmonic components. Next, the two parameters of interest are estimated using the least squares method given the phases of each component.

In order to separate the components, the quadratic term in the phase must be eliminated. This is termed as de-chirping process since eliminating the quadratic term yields a sinusoid in the presence of other harmonic components and noise. Therefore is can be easily filtered using standard filtering methods. We define a de-chirping set $\Omega = \{\theta_{2,1}, \ldots, \theta_{2,L}\}$ of L frequency rate candidates. For each candidate, M dechirped signals are defined

$$\begin{aligned} \mathbf{x}_{\ell,m} &= \mathbf{D}(m\theta_{2,\ell})\mathbf{x},\ m = 1, \ldots, M \qquad (12) \\ \mathbf{D}(m\theta_{2,\ell}) &= \operatorname{diag}(1, \ldots, e^{-j2\pi\frac{1}{2}m\theta_{2,\ell}(N-1)^2}). \qquad (13) \end{aligned}$$

Denote by $\bar{\mathbf{x}}_{\ell,m}$ the discrete Fourier transform (DFT) of $\mathbf{x}_{\ell,m}$

$$\bar{\mathbf{x}}_{\ell,m} = \mathbf{W}\mathbf{x}_{\ell,m} \overset{\text{by (12)}}{=} \mathbf{W}\mathbf{D}(m\theta_{2,\ell})\mathbf{x} \qquad (14)$$

where $\mathbf{W}$ is the $N \times N$ DFT transform matrix.

When the candidate frequency rate, $\theta_{2,\ell}$, corresponds to the true frequency rate, each $\bar{\mathbf{x}}_{\ell,m}$ should have a strong peak, with a value of $N|a_m|$, at the frequency corresponding to the true initial frequency. Hence a suitable candidate can be found by summing the maximum value of each component, i.e.,

$$\tilde{\theta}_2 = \underset{\theta_{2,\ell}\in\Omega}{\operatorname{argmax}} \sum_{m=1}^{M} |\bar{x}_{\ell,m}[k_m^{(\max)}(\theta_{2,\ell})]| \qquad (15)$$

where $\bar{x}_{\ell,m}[k_{m,\ell}^{(\max)}]$ is the maximum value of $\bar{\mathbf{x}}_{\ell,m}[k]$, and the location of the maximum is,

$$k_{m,\ell}^{(\max)} = \underset{k}{\operatorname{argmax}} |\bar{x}_{\ell,m}[k]|. \qquad (16)$$

Other possible selection criteria can be based on computing the sparsity of $\bar{\mathbf{x}}_{\ell,m}$, e.g., using its Kurtosis [17].

Given $\tilde{\theta}_2 = \theta_{2,\tilde{\ell}}$, each harmonic components can be separated in the frequency domain by a bandpass filter with a width of Δ as follows

$$\mathbf{Y}_m = \operatorname{diag}\left([\mathbf{0}^T_{k^{(\max)}_{m,\tilde{\ell}}-\Delta/2}, \mathbf{1}^T_{\Delta}, \mathbf{0}^T_{N-k^{(\max)}_{m,\tilde{\ell}}-\Delta/2}]^T\right) \qquad (17)$$

where $\mathbf{1}_N$ ($\mathbf{0}_N$) is an $N \times 1$ vector with all elements equal to one (zero). The filtered signal in the time domain is obtained by performing an inverse DFT (IDFT) followed by a chirp multiplication, i.e.,

$$\begin{aligned} \hat{\mathbf{s}}_m &= \mathbf{D}(m\tilde{\theta}_2)^H\mathbf{W}^H\mathbf{Y}_m\bar{\mathbf{x}}_{\ell,m} \\ &\overset{\text{by (14)}}{=} \mathbf{D}(m\tilde{\theta}_2)^H\mathbf{W}^H\mathbf{Y}_m\mathbf{W}\mathbf{D}_m(m\tilde{\theta}_2)\mathbf{x}. \qquad (18) \end{aligned}$$

We thus obtain a set of M harmonic components, $\{\hat{\mathbf{s}}_1, \ldots, \hat{\mathbf{s}}_M\}$, of the observed signal.

Once the harmonic components are separated, a joint phase unwrapping and parameter estimation recursive process [7] is used to estimate the parameters of the fundamental chirp. The phase of the mth component is given by

$$\tilde{\phi}_m[n] \cong \mu_m + 2\pi m(\theta_1 n + \frac{1}{2}\theta_2 n^2) + \varepsilon_m[n] \qquad (19)$$

where $\varepsilon_m[n]$ is an error caused by the filtered noise. Define the vector of phase measurements obtained from the mth reconstructed harmonic component, $\hat{\mathbf{s}}_m$, by $\tilde{\boldsymbol{\phi}}_m \triangleq [\tilde{\phi}_m[0], \ldots, \tilde{\phi}_m[N-1]]^T$,

$$\tilde{\boldsymbol{\phi}}_m = \mu_m\mathbf{1}_N + 2\pi m\mathbf{H}\boldsymbol{\theta} + \boldsymbol{\varepsilon}_m\,, m = 1, \ldots, M \qquad (20)$$

where $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2]$ with $\mathbf{h}_1 = [0, 1, \ldots, N-1]^T$, and $\mathbf{h}_2 = [0^2/2, 1^2/2, \ldots, (N-1)^2/2]^T$.

We now present the recursive estimation process. At each step, $\hat{\boldsymbol{\eta}}[n] = [\hat{\boldsymbol{\mu}}^T[n],\ \hat{\boldsymbol{\theta}}^T[n]]^T$ is a vector of the current estimates of the parameters, where $\hat{\boldsymbol{\mu}}[n] = [\hat{\mu}_1[n], \ldots, \hat{\mu}_M[n]]^T$. Then, the unwrapped phase is given by

$$\tilde{\boldsymbol{\phi}}[n] = \mathbf{H}[n]\hat{\boldsymbol{\eta}}[n] + \boldsymbol{\varepsilon}[n] \qquad (21)$$

where

$$\mathbf{H}[n] = \left[\begin{array}{c|cc} & 2\pi n & \pi n^2 \\ \mathbf{I}_M & \vdots & \vdots \\ & 2\pi M n & \pi M n^2 \end{array}\right] \qquad (22)$$

$\tilde{\boldsymbol{\phi}}[n] = [\tilde{\phi}_1[n], \ldots, \tilde{\phi}_M[n]]^T$, $\boldsymbol{\varepsilon}[n] = [\varepsilon_1[n], \ldots, \varepsilon_M[n]]^T$ and $\mathbf{I}_M$ is the $M \times M$ identity matrix.

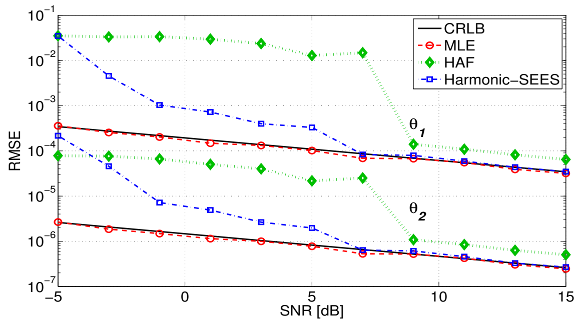


Fig. 1: RMSE for each estimator including the CRLB.

The algorithm is initialized with an estimate given $L \geq M+2$ samples of the unwrapped phased using a conventional unwrapping algorithm, e.g. [6],

$$\hat{\boldsymbol{\eta}}[L] = \left(\mathbf{H}_L^T \mathbf{H}_L\right)^{-1} \mathbf{H}_L^T \tilde{\boldsymbol{\phi}}_L \quad (23)$$

where $\mathbf{H}_L = [\mathbf{H}^T[0], \ldots, \mathbf{H}^T[L-1]]^T$ and $\tilde{\boldsymbol{\phi}}_L = [\tilde{\boldsymbol{\phi}}^T[0], \ldots, \tilde{\boldsymbol{\phi}}^T[L-1]]^T$ is a vector of the unwrapped phases of all harmonic components up to the L'th step. For the nth step, where $n = L+1, \ldots, N-1$, the following is performed. First, the unwrapped phases are predicted using

$$\tilde{\boldsymbol{\phi}}[n|n-1] = \mathbf{H}[n]\hat{\boldsymbol{\eta}}[n-1]. \quad (24)$$

Next, the actual unwrapped phases are given by

$$\tilde{\phi}_m[n+1] = \arg\left(\hat{s}_m[n] e^{-j\tilde{\phi}_m[n|n-1]}\right) + \tilde{\phi}_m[n|n-1]. \quad (25)$$

Finally, the estimated vector, $\hat{\boldsymbol{\eta}}$, is updated,

$$\hat{\boldsymbol{\eta}}[n+1] = \hat{\boldsymbol{\eta}}[n] + \mathrm{K}_{m+1}\left(\tilde{\boldsymbol{\phi}}[n+1] - \mathbf{H}[n+1]\hat{\boldsymbol{\eta}}[n]\right) \quad (26)$$

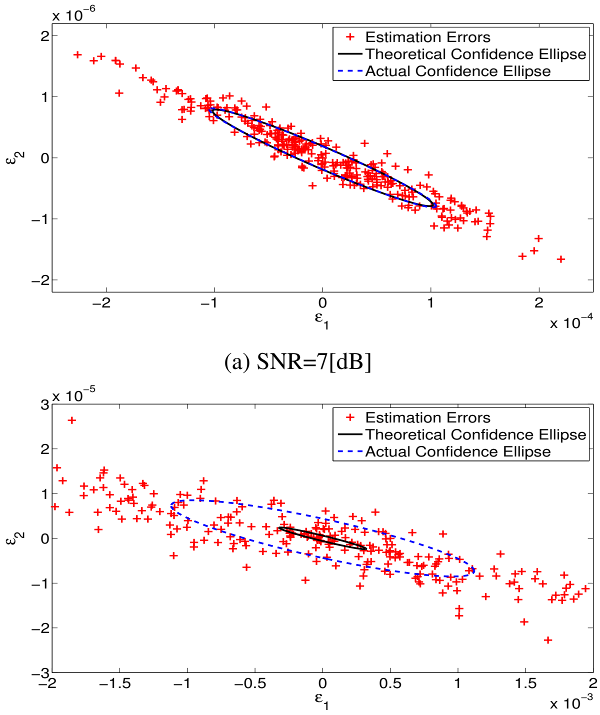
where $\mathrm{K}_{m+1} = \left(\mathbf{H}_{n+1}^T \mathbf{H}_{n+1}\right)^{-1} \mathbf{H}^T[n+1]$. Note that K_{m+1} is not data dependent and can be computed off-line. The estimated parameters are obtained from the final step, $\hat{\boldsymbol{\theta}}^{(\mathrm{HSEES})} = \hat{\boldsymbol{\theta}}[N-1]$.

Due to the use of the DFT rather than the DTFT, it can be shown that the computational complexity of the Harmonic-SEES is $\mathcal{O}\left(MN^2 \log N\right)$ [14]. Unlike the HAF method, it is proportional to the number of harmonic components. However, this number is usually small and therefore the Harmonic-SEES method performs better than the HAF method in terms of computational complexity.

6. EXPERIMENTAL RESULTS

We now compare the performance of the MLE, the HAF method and the Harmonic-SEES method. We consider $M = 4$ harmonics with initial frequency and frequency rate of $\theta_1 = 0.15$ and $\theta_2 = -2.5 \cdot 10^{-4}$, respectively. The amplitude of the mth harmonic is given by $a_m = 2^{(1-m)/2} e^{j\mu_m}$, where $\{\mu_m\}_{m=1}^M$ are uniform i.i.d in the range of $[0, 2\pi]$ and were generated once per scenario. The number of samples is $N = 256$. The noise power, σ_v^2, is adjusted to give the desired SNR defined as $\mathrm{SNR} = 10\log_{10}\left(\sum_{m=1}^M |a_m|^2/\sigma_v^2\right)$ [dB].

First we evaluate the performances of the estimators in terms of root mean squared error (RMSE) defined as $\mathrm{RMSE}(\theta_k) = \sqrt{\frac{1}{Nexp}\sum_{i=1}^{N_{exp}} \varepsilon_{i,k}^2}$, $k = 1, 2$ where $\varepsilon_{i,k} = \hat{\theta}_{k,i} - \theta_k$ is the estimation error and $\hat{\theta}_{k,i}$ is the estimate of θ_k at the ith trial. N_{exp} = 300 is the number of Monte-Carlo independent trials. We consider SNR values ranging from -5[dB] to 15[dB]. The RMSE versus SNR results for both parameters are presented in Fig. 1. The CRLB is also plotted. The Harmonic-SEES estimates of both parameters achieve the CRLB for SNR values of 7[dB] or more. The HAF based estimation does not achieves the CRLB. However, it obtains a good estimation for SNR values of 9[dB] or more.



(a) SNR=7[dB]

(b) SNR=-3[dB]

Fig. 2: Estimation errors and confidence ellipses of the fundamental frequency and frequency rate.

The scattering of the estimation errors for the Harmonic-SEES method are presented in Fig. 2 for SNR of 7[dB] and -3[dB]. In addition, in each plot, the theoretical and actual 50% confidence ellipses are presented. The theoretical ellipse is calculated from the CRLB and the actual ellipse is calculated from the covariance of the results. For SNR of -3[dB], the RMSE of the Harmonic-SEES method is much higher than the theoretical bound. Therefore the actual ellipse in that case is much larger than the theoretical. For SNR

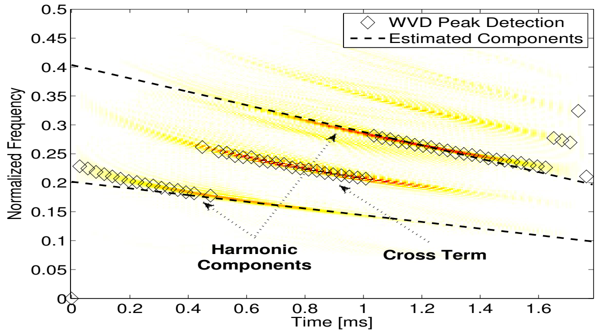


Fig. 3: Wigner-Ville distribution and estimated harmonic components of an *E. Fuscus* bat call.

of 7[dB], where the Harmonic-SEES achieves the theoretical bound, both ellipses are similar.

Next we demonstrate the parameter estimation of an echolocation call produced by an *E. Fuscus* bat [18]. We used 256 samples to estimate the parameters of $M = 2$ harmonic components. Figure 3 presents the estimated components plotted on the Wigner-Ville distribution (WVD) of the signal. Two harmonic component can be seen, along with a cross-term between them. Note that the components are not linearly modulated. The markers on the WVD denote local peak detection. It can be seen that the cross-term is in part stronger than the harmonic components.

7. CONCLUSION

We addressed the problem of estimating the parameters of harmonic linear chirps. The MLE for the parameters of interest requires a high resolution two-dimensional search and therefore involves a large number of computations. We suggested a sub-optimal low complexity estimation method, namely the Harmonic-SEES, that first separates the harmonic components and then the parameters are estimated using a joint least squares given the phases of the separated components. We also considered the HAF based estimation method with modifications for the case of harmonic components. Simulations show that the Harmonic-SEES method achieves the Cramer-Rao lower bound at high SNR and performs better than the HAF in terms of RMSE.

REFERENCES

[1] M. Bennett, S. McLauglin, T. Anderson, and N. McDicken, "Filtering of chirped ultrasound echo signals with the fractional Fourier transform," in *IEEE Ultrasonics Symposium*, Aug. 2004, vol. 3, pp. 2036–2040.

[2] M. Vespe, G. Jones, and C.J. Baker, "Lessons for radar: waveform diversity in echolocating mammals," *IEEE Signal Process. Mag.*, vol. 26, no. 1, pp. 65–75, Jan. 2009.

[3] T. H. Chung and J.Y. Cheung, "Maximum likelihood estimation of direction of arrival and frequency sweeping rate with linear chirp signals," *IEEE Signal Process. Lett.*, vol. 2, no. 8, pp. 163–165, Aug. 1995.

[4] R. Aouada, A. Belouchrani, and K. Abed-Meraim, "Multipath parameter estimation of linear chirp signals using sensor arrays," in *Proc. IEEE-SAM*, July 2004, pp. 313–317.

[5] R. Kumaresan and S. Verma, "On estimating the parameters of chirp signals using rank reduction techniques," in *Proc. Asilomar Conf. Signals, Syst. Comput.* Pacific Grove, CA, 1987, pp. 555–558.

[6] P. M. Djuric and S. M. Kay, "Parameter estimation of chirp signals," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, no. 12, pp. 2118–2126, Dec. 1990.

[7] B. J. Slocumb and J. Kitchen, "A polynomial phase parameter estimation-phase unwrapping algorithm," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*. Adelaide, Australia, Apr. 1994, vol. 4, pp. 129–132.

[8] S. Peleg and B. Porat, "Estimation and classification of polynomial-phase signals," *IEEE Trans. Inf. Theory*, vol. 37, no. 2, pp. 422–430, Mar. 1991.

[9] S. Peleg and B. Friedlander, "Multicomponent signal analysis using the polynomial-phase transform," *IEEE Trans. Aerosp. and Electron. Syst.*, vol. 32, no. 1, pp. 378–387, Jan. 1996.

[10] S. Barbarossa, "Detection and estimation of the instantaneous frequency of polynomial-phase signals by multilinear time-frequency representations," in *Proc. IEEE Signal Process. Workshop Higher Order Stat.* Lake Tahoe, CA, June 1993, pp. 168–172.

[11] M. Farquharson, P. O'Shea, and G. Ledwich, "A computationally efficient technique for estimating the parameters of polynomial-phase signals from noisy observations," *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 3337–3342, Aug. 2005.

[12] D. S. Pham and A. M. Zoubir, "Analysis of multicomponent polynomial phase signals," *IEEE Trans. Signal Process.*, vol. 55, no. 1, pp. 56–65, Jan. 2007.

[13] Y. Wu, H. C. So, and H. Liu, "Subspace-based algorithm for parameter estimation of polynomial phase signals," *IEEE Trans. Signal Process.*, vol. 56, no. 10, pp. 4977–4983, Oct. 2008.

[14] Y. Doweck, A. Amar, and I. Cohen, "Joint model order selection and parameter estimation of chirps with harmonic components," *IEEE Trans. Signal Process.*, vol. 63, no. 7, pp. 1765–1778, Apr. 2015.

[15] A. Amar, A. Leshem, and A. J. van der Veen, "A low complexity blind estimator of narrowband polynomial phase signals," *IEEE Trans. Signal Process.*, vol. 58, no. 9, pp. 4674–4683, Sept. 2010.

[16] H.L. Van Trees, *Detection, Estimation, and Modulation Theory - Part IV, Optimum Array Processing*, Wiley, 2004.

[17] N. Hurley and R Scott, "Comparing measures of sparsity," *IEEE Trans. Inf. Theory*, vol. 55, no. 10, pp. 4723–4741, Oct. 2009.

[18] C. Condon, K. White, and A. Feng, "Bat echolocation chirp," [Online]. Available: http://dsp.rice.edu/software/bat-echolocation-chirp.